Optimization in Engineering Applications

Static Analysis – Truss Framework

A simple truss structure, shown to the right, shall be optimized. The truss consists of two round bars with Young’s modulus $E = 2.1 \cdot 10^{11}$ N/m$^2$ and density $\rho = 2750$ kg/m$^3$. As design variables the radii of the bars $r_1$ and $r_2$ are chosen

$$ p = [r_1 \, r_2] , \text{ whereby } 2 \text{ mm} \leq r_i \leq 5 \text{ mm}, \ i = 1,2. $$

When applying a force $F = 100$ N at point $B$ a displacement $u$ is caused, which can be computed using the finite element method

$$ Ku = q, $$

with the stiffness matrix

$$ K = \frac{E}{\ell^2 \sqrt{2}} \begin{bmatrix} A_2 & A_1 \frac{2\sqrt{2}}{} + A_2 \\ A_2 & A_2 \\ A_2 & A_2 \end{bmatrix}, $$

the vector of nodal coordinates $u = [u_x \, u_y] \top$ and the vector of applied forces $q = [0 \, F] \top$. In an optimization the displacement $u_y$ shall be minimized. Thus, the scalar objective function reads

$$ \psi(p) = u_y = \frac{\sqrt{2} F\ell}{2 E} \left( \frac{4r_1^2 + \sqrt{2}r_2^2}{\pi r_1^2 r_2^2} \right). $$

Evaluating $\psi(p)$ in the feasible design space returns the following results.

It can be seen that by increasing the radii, the displacement is reduced. Thus, if there are no additional constraint equations, such as mass restriction, the solution of the minimization problem is $p_1^* = p_2^* = 5$ mm and $\psi(p^*) = 0.07$ mm.
Dynamic Analysis – Slider-Crank Mechanism

Not only static but also dynamic problems are analyzed and optimized in engineering. For instance, using the method of multibody systems the slider-crank mechanism shown below is modeled. The multibody systems consists of the crank ($m_1 = 0.24 \text{kg}$, $J_1 = 0.26 \text{ kg m}^2$), the piston rod ($m_2 = 0.16 \text{ kg}$, $J_2 = 0.0016 \text{ kg m}^2$) as well as the slider block ($m_3 = 0.46 \text{ kg}$). The crank angle is assumed to rotate at constant angular velocity $\omega = 8 \text{ Hz}$ and, thus, the motion of the mechanism is clearly defined.

Performing a simulation for the time domain $t \in [0 \ 3] \text{ s}$, the resulting reaction force between the crank and the inertial frame, which is defined as

$$ F(p, t) = \sqrt{F_x^2(p, t) + F_y^2(p, t)}, $$

can be computed. For two different values $p = -0.02 \text{ m}$ and $p = -0.03 \text{ m}$ the resulting reaction forces $F(p, t)$ are displayed below.

Performing an optimization, $F(p, t)$ shall be minimized. However, in contrast to static problems, first the transient system response has to be converted into a scalar value. Therefore, the time-dependent resulting reaction force $F(p, t)$ is integrated over the simulation time $t$. Thus, it holds for the objective function

$$ \psi(p) = \int_{t_0}^{t_f} F(p, t) \, dt = \int_{0}^{3s} \sqrt{F_x^2(p, t) + F_y^2(p, t)} \, dt. $$
Then, evaluating the objective function \( \psi(p) \) for \( p \in [-0.02, -0.01] \) m the local minimum can be determined as \( p^* \approx -0.017 \) and \( \psi(p^*) \approx 0.646 \).

### Dynamic Analysis – Planar 2-Arm Welding Robot

A further example for the optimization of dynamic systems is the planar 2-arm welding robot shown below. For the welding process the tool center point (TCP) has to follow a semi-circular trajectory (—) within 3 seconds. The joint angles \( \varphi \) and \( \psi \) are modeled as rheonomic constraints, i.e., \( \varphi = \varphi(t) \) and \( \psi = \psi(t) \). However, due to joint elasticity, which is modeled by rotational springs with stiffness \( c \), there are additional rotations of the two arms \( \Delta \varphi \) and \( \Delta \psi \). These additional rotations represent the generalized degrees of freedom of the system \( y = [\Delta \varphi \quad \Delta \psi]^T \). As a consequence, the actual trajectory of the TCP (---) differs from the desired trajectory.
By varying the design variables $p$ the center of gravity of the second arm is changed and, thereby, the tracking error of the TCP shall be reduced. The tracking error $F$ is determined by the Euclidean distance between the actual position $r_a = [x_a \ y_a]^T$ and the desired position $r_d = [x_d \ y_d]^T$ and is computed as

$$F(p, y, t) = \sqrt{(x_a(p, y, t) - x_d(t))^2 + (y_a(p, y, t) - y_d(t))^2}.$$  

It can be seen, that not only the tracking error $F$ but also that the generalized degrees of freedom $\Delta \varphi$ and $\Delta \psi$ depend on the design variable $p$.

To obtain a scalar objective function, the tracking error $F$ is integrated over the simulation time

$$\psi(p) = \int_0^{3s} F(p, y, t) \, dt = \int_0^{3s} \sqrt{(x_a(p, y, t) - x_d(t))^2 + (y_a(p, y, t) - y_d(t))^2} \, dt.$$  

Evaluating the objective function for $p \in [-0.02 \ -0.01]^T$, a local minimum can be graphically determined at $p^* \approx -0.04$ and $\psi(p^*) \approx 0.0039$. 