Matrizenalgebra

Matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}: \boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad a_{ij} \in \mathbb{R},$

Vektor $\boldsymbol{x} \in \mathbb{R}^{n}: \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad x_i \in \mathbb{R}.$

Elementare Operationen

Operation	Schreibweise	Koordinaten	Abbildung
Addition	$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$	$c_{ij} = a_{ij} + b_{ij}$	$\mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$
Multiplikation mit Skalar	$\boldsymbol{C} = \alpha\boldsymbol{A}$	$c_{ij} = \alpha a_{ij}$	$\mathbb{R} \times \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$
Transponieren	$\boldsymbol{C} = \boldsymbol{A}^{\mathrm{T}}$	$c_{ij} = a_{ji}$	$\mathbb{R}^{m\times n} \to \mathbb{R}^{n\times m}$
Differentiation	$\boldsymbol{C} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}$	$c = \frac{\mathrm{d}}{\mathrm{d}t} a_{ij}$	$\mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$
Matrizenmultiplikation	$\boldsymbol{y} = \boldsymbol{A} \cdot \boldsymbol{x}$ $\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B}$	$y_i = \sum_k a_{ik} x_k$ $C_{ij} = \sum_k a_{ik} b_{kj}$	$\mathbb{R}^{m\times n} \times \mathbb{R}^{n} \to \mathbb{R}^{m}$ $\mathbb{R}^{m\times n} \times \mathbb{R}^{n\times p} \to \mathbb{R}^{m\times p}$
Inneres Produkt (Skalarprodukt)	$\alpha = \boldsymbol{x} \cdot \boldsymbol{y}$	$\alpha = \sum_k x_k y_k$	$\mathbb{R}^{n} \times \mathbb{R}^{n} \to \mathbb{R}$
Äußeres Produkt (Dyadisches Produkt)	$\boldsymbol{A} = \boldsymbol{x}\boldsymbol{y}$	$a_{ij} = x_i y_j$	$\mathbb{R}^{m} \times \mathbb{R}^{n} \to \mathbb{R}^{m\times n}$

Rechenregeln

Addition:
$$\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$$
$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$$

Multiplikation mit Skalar:
$$\alpha(\boldsymbol{A} \cdot \boldsymbol{B}) = (\alpha\boldsymbol{A}) \cdot \boldsymbol{B} = \boldsymbol{A} \cdot (\alpha\boldsymbol{B})$$
$$\alpha(\boldsymbol{A} + \boldsymbol{B}) = \alpha\boldsymbol{A} + \alpha\boldsymbol{B}$$
$$(\alpha + \beta)\boldsymbol{A} = \alpha\boldsymbol{A} + \beta\boldsymbol{A}$$

Transposition:
$$(\boldsymbol{A}^T)^T = \boldsymbol{A}$$
$$(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$$
$$(\alpha\boldsymbol{A})^T = \alpha\boldsymbol{A}^T$$
$$(\boldsymbol{A} \cdot \boldsymbol{B})^T = \boldsymbol{B}^T \cdot \boldsymbol{A}^T$$

Differentiation:
$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{A} + \boldsymbol{B}) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A} + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{B}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{A} \cdot \boldsymbol{B}) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}\right) \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{B}\right)$$

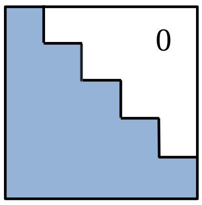
Matrizenmultiplikation:
$$\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$$
$$\boldsymbol{A} \cdot (\boldsymbol{B} \cdot \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{B}) \cdot \boldsymbol{C}$$
$$\boldsymbol{A} \cdot \boldsymbol{B} \neq \boldsymbol{B} \cdot \boldsymbol{A} \quad \text{im Allgemeinen}$$

Skalarprodukt:
$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$$
$$\boldsymbol{x} \cdot \boldsymbol{x} \geq 0 \quad \forall \boldsymbol{x}, \quad \boldsymbol{x} \cdot \boldsymbol{x} = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{0}$$
$$\boldsymbol{x} \cdot \boldsymbol{y} = 0 \Leftrightarrow \boldsymbol{x}, \boldsymbol{y} \text{ orthogonal}$$

Spezielle quadratische Matrizen

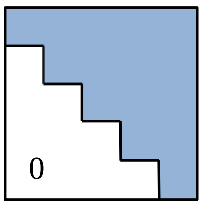
Linksdreiecksmatrix, untere Dreiecksmatrix
(lower triangular $\boldsymbol{L}$)

$\boldsymbol{L} =$



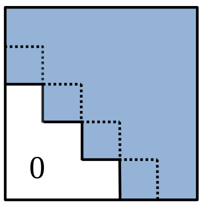
Rechtsdreiecksmatrix, obere Dreiecksmatrix
(upper triangular $\boldsymbol{R}$)

$\boldsymbol{R} =$



(obere) Hessenberg–Matrix

$\boldsymbol{H} =$



Einheitsmatrix
$$\boldsymbol{E} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Diagonalmatrix

$$\boldsymbol{D}\cdot\boldsymbol{A}=\boldsymbol{D}\cdot\begin{bmatrix}\overline{\boldsymbol{a}}_1\\ \vdots\\ \overline{\boldsymbol{a}}_n\end{bmatrix}=\begin{bmatrix}d_1\overline{\boldsymbol{a}}_1\\ \vdots\\ d_n\overline{\boldsymbol{a}}_n\end{bmatrix},\quad \boldsymbol{D}^{-1}=\mathrm{diag}\left\{\frac{1}{d_i}\right\}\text{ für } d_i\neq 0$$

$$\boldsymbol{A}\cdot\boldsymbol{D}=[\boldsymbol{a}_1\ \dots\ \boldsymbol{a}_n]\cdot\boldsymbol{D}=[d_1\boldsymbol{a}_1\ \dots\ d_n\boldsymbol{a}_n]$$

Permutationsmatrix

$$\boldsymbol{P}_{rs}\cdot\boldsymbol{A}=\boldsymbol{P}_{rs}\cdot\begin{bmatrix}\vdots\\ \overline{\boldsymbol{a}}_r\\ \vdots\\ \overline{\boldsymbol{a}}_s\\ \vdots\end{bmatrix}=\begin{bmatrix}\vdots\\ \overline{\boldsymbol{a}}_s\\ \vdots\\ \overline{\boldsymbol{a}}_r\\ \vdots\end{bmatrix},\quad \boldsymbol{P}_{rs}=\begin{bmatrix}1 &&&&&&&&&\\ &\ddots&&&&&&&&\\ &&1&&&&&&&\\ &&&0&&&1&&&\\ &&&&1&&&&&\\ &&&&&\ddots&&&&\\ &&&&&&1&&&\\ &&&1&&&0&&&\\ &&&&&&&1&&\\ &&&&&&&&\ddots&\\ &&&&&&&&&1\end{bmatrix}\begin{matrix}\\ \\ \\ r\\ \\ \\ \\ s\\ \\ \\ \\ \end{matrix}$$

$$\qquad\qquad\qquad\qquad\qquad r \qquad\qquad\qquad s$$

$$\boldsymbol{A}\cdot\boldsymbol{P}_{rs}=[\dots\ \boldsymbol{a}_r\ \dots\ \boldsymbol{a}_s\ \dots]\cdot\boldsymbol{P}_{rs}=[\dots\ \boldsymbol{a}_s\ \dots\ \boldsymbol{a}_r\ \dots]$$

$\boldsymbol{P}_{rs}^T=\boldsymbol{P}_{rs}$ symmetrisch
$\boldsymbol{P}_{rs}^{-1}=\boldsymbol{P}_{rs}^T$ orthogonal
$\Rightarrow \boldsymbol{P}_{rs}^2=\boldsymbol{E}$ involutorisch
Allgemein: $\boldsymbol{P}=[\boldsymbol{e}_{s_1}\ \dots\ \boldsymbol{e}_{s_n}]$ mit
$\boldsymbol{e}_i$ Einsvektoren
$(s_1,\dots,s_n)$: Permutation von $(1,\dots,n)$

Inverse Matrix $\boldsymbol{A}^{-1}$	$\boldsymbol{A}^{-1}\cdot\boldsymbol{A}=\boldsymbol{A}\cdot\boldsymbol{A}^{-1}=\boldsymbol{E}$
	$(\boldsymbol{A}\cdot\boldsymbol{B})^{-1}=\boldsymbol{B}^{-1}\cdot\boldsymbol{A}^{-1}$
Symmetrische Matrix	$\boldsymbol{A}=\boldsymbol{A}^T$
Schiefsymmetrische Matrix	$\boldsymbol{A}=-\boldsymbol{A}^T$
Positiv definite Matrix	$\boldsymbol{x}\cdot\boldsymbol{A}\cdot\boldsymbol{x}>0\quad\forall\boldsymbol{x}\neq\boldsymbol{0}$
Orthogonale Matrix $(\boldsymbol{A}\cdot\boldsymbol{A}^T=\boldsymbol{E})$	$\boldsymbol{A}^{-1}=\boldsymbol{A}^T$
Involutorische Matrix	$\boldsymbol{A}^2=\boldsymbol{E}$
Diagonaldominante Matrix	$\lvert a_{ii}\rvert>\sum_{j\neq i}\lvert a_{ij}\rvert\quad\forall i$