Verfahren von Householder

Householder–Transformation

Gesucht ist eine Matrix $\boldsymbol{H} \in \mathbb{R}^{m \times m}$, für die gilt:

(i) $\boldsymbol{H}$ ist orthogonal,

(ii) $\boldsymbol{H} \cdot \boldsymbol{a} = \mu \boldsymbol{e}_1, \quad \boldsymbol{a} = [a_1, \dots, a_m]^T, \quad \mu \in \mathbb{R}.$

Ansatz: $$\boldsymbol{H} := \boldsymbol{E} - 2\boldsymbol{u} \cdot \boldsymbol{u}^{\boldsymbol{T}}, \quad \|\boldsymbol{u}\|_2 = 1;$$

Aus (i): $\boldsymbol{H} \cdot \boldsymbol{H}^T =$ _____

Aus (ii) einerseits:

$$\|\boldsymbol{H} \cdot \boldsymbol{a}\|_2 = \sqrt{\boldsymbol{a}^T \cdot \boldsymbol{H}^T \cdot \boldsymbol{H} \cdot \boldsymbol{a}} \equiv \|\boldsymbol{a}\|_2 \overset{!}{=} \sqrt{\mu \cdot \boldsymbol{e}_1^T \cdot \boldsymbol{e}_1 \cdot \mu} = |\mu|$$

$\rightarrow \mu =$ _____

Aus (ii) andererseits:

$$\boldsymbol{H} \cdot \boldsymbol{a} = \underline{\qquad\qquad\qquad} \overset{!}{=} \mu \boldsymbol{e}_1$$

$$\rightarrow (2\boldsymbol{u}^{\boldsymbol{T}} \cdot \boldsymbol{a}) \cdot \boldsymbol{u} = \boldsymbol{a} - \mu \boldsymbol{e}_1 = \boldsymbol{v}$$

$$\rightarrow \boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|_2}.$$

$\Longrightarrow \boldsymbol{v} = \boldsymbol{a} - \mu \boldsymbol{e}_1 =$ _____

Damit bleibt:

1) $\mu = -\mathrm{sign}(a_1) \cdot \|\boldsymbol{a}\|_2$

2) $\boldsymbol{v} = \boldsymbol{a} - \mu \boldsymbol{e}_1 =$ _

3) $\alpha =$ _ _ _ _ _ _ _ _ _ _ _

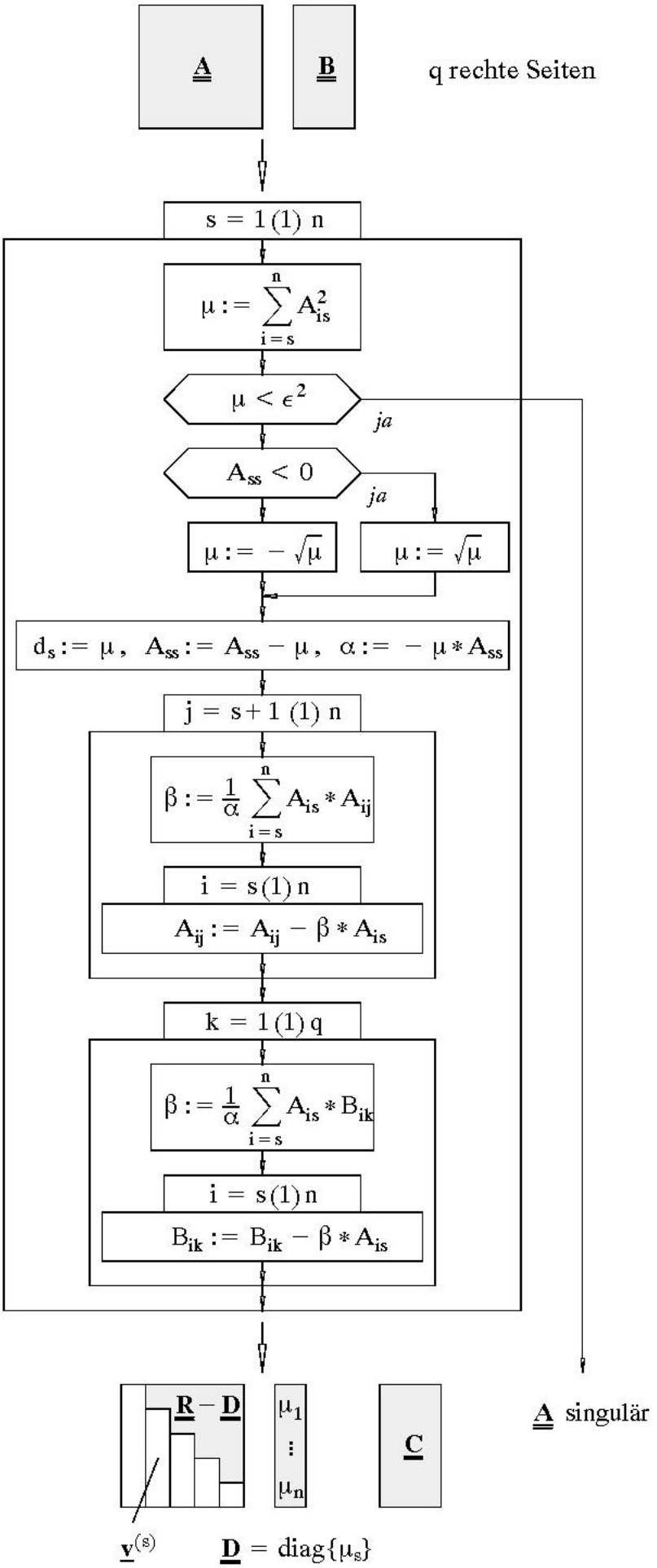
4) $\boldsymbol{H} = \boldsymbol{E} - 2\boldsymbol{u} \cdot \boldsymbol{u}^T =$ _

Transformation eines Vektors $\boldsymbol{x} \in \mathbb{R}^m$:

$$\boldsymbol{H} \cdot \boldsymbol{x} := \boldsymbol{x} - \frac{\boldsymbol{v}^T \cdot \boldsymbol{x}}{\alpha} \boldsymbol{v}$$

Algorithmus

$\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{B}$,

$\boldsymbol{A} \in \mathbb{R}^{n \times n}$
$\boldsymbol{X}, \boldsymbol{B} \in \mathbb{R}^{n \times q}$

Rechenaufwand
$\approx \frac{2}{3} n^3 + q n^2$ flops