Lagrange'sche Gleichungen zweiter Art

Prinzip von d'Alembert: $$\sum_k \left(m_k \mathbf{a}_k^T - \mathbf{f}_k^{eT}\right) \cdot \delta \mathbf{r}_k = 0$$

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$$\sum_k \left[\left(m_k \dot{\mathbf{v}}_k^T \cdot \dot{\mathbf{v}}_k - \mathbf{f}_k^{eT}\right) \cdot \sum_{i=1}^{f} \frac{\partial \mathbf{r}_k}{\partial y_i} \delta y_i\right] = \sum_{i=1}^{f} \left[\sum_k m_k \dot{\mathbf{v}}_k^T \cdot \frac{\partial \mathbf{r}_k}{\partial y_i} - \sum_k \mathbf{f}_k^{eT} \cdot \frac{\partial \mathbf{r}_k}{\partial y_i}\right] \delta y_i = 0$$

- $$\mathbf{v}_k = \frac{d\mathbf{r}_k}{dt} = \sum_i \frac{\partial \mathbf{r}_k}{\partial y_i} \dot{y}_i \Rightarrow \frac{\partial \mathbf{v}_k}{\partial \dot{y}_i} = \frac{\partial \mathbf{r}_k}{\partial y_i}$$
- $$\frac{\partial}{\partial y_i}\left(\mathbf{v}_k^T \cdot \mathbf{v}_k\right) = \frac{\partial \mathbf{v}_k^T}{\partial y_i} \cdot \mathbf{v}_k + \mathbf{v}_k^T \cdot \frac{\partial \mathbf{v}_k}{\partial y_i} = 2\, \mathbf{v}_k^T \cdot \frac{\partial \mathbf{v}_k}{\partial y_i}$$
- $$m_k \dot{\mathbf{v}}_k^T \cdot \frac{\partial \mathbf{r}_k}{\partial y_i} = \frac{d}{dt}\left[m_k \mathbf{v}_k^T \cdot \frac{\partial \mathbf{r}_k}{\partial y_i}\right] - m_k \mathbf{v}_k^T \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_k}{\partial y_i}\right)$$
$$= \frac{d}{dt}\left[m_k \mathbf{v}_k^T \cdot \frac{\partial \mathbf{v}_k}{\partial \dot{y}_i}\right] - m_k \mathbf{v}_k^T \cdot \frac{\partial}{\partial y_i}\left(\frac{d\mathbf{r}_k}{dt}\right)$$
$$= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{y}_i}\left(\frac{1}{2} m_k \mathbf{v}_k^T \cdot \mathbf{v}_k\right)\right] - \frac{\partial}{\partial y_i}\left(\frac{1}{2} m_k \mathbf{v}_k^T \cdot \mathbf{v}_k\right)$$
$$= \frac{d}{dt}\frac{\partial T_k}{\partial \dot{y}_i} - \frac{\partial T_k}{\partial y_i}$$
- $$T = \sum_k T_k = \sum_k \frac{1}{2} m_k \mathbf{v}_k^T \cdot \mathbf{v}_k = \sum_k \frac{1}{2} m_k v_k^2$$ kinetische Energie
- $$Q_i = \sum_k \mathbf{f}_k^{eT} \cdot \frac{\partial \mathbf{r}_k}{\partial y_i}$$ verallgemeinerte Kraft zum Freiheitsgrad y_i

$$\sum_{i=1}^{f}\left[\frac{d}{dt}\frac{\partial T}{\partial \dot{y}_i} - \frac{\partial T}{\partial y_i} - Q_i\right] \delta y_i = 0$$

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Lagrange Gleichungen zweiter Art

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}_i} - \frac{\partial T}{\partial y_i} = Q_i\,, \quad i = 1(1)f$$

speziell: konservative Systeme

eingeprägte Kräfte haben ein Potential $\mathrm{U_k}(\mathbf{r}_\mathrm{k})$ mit $\mathbf{f}_\mathrm{k}^\mathrm{e} = -\mathrm{grad}\,\mathrm{U_k} = -\dfrac{\partial \mathrm{U_k}}{\partial \mathbf{r}_\mathrm{k}}$

$$\mathrm{Q_i} = \sum_\mathrm{k} \mathbf{f}_\mathrm{k}^{\mathrm{e}\,\mathrm{T}} \cdot \frac{\partial \mathbf{r}_\mathrm{k}}{\partial \mathrm{y_i}} = -\sum_\mathrm{k} \left(\frac{\partial \mathrm{U_k}}{\partial \mathbf{r}_\mathrm{k}}\right)^\mathrm{T} \cdot \frac{\partial \mathbf{r}_\mathrm{k}}{\partial \mathrm{y_i}} = -\sum_\mathrm{k} \frac{\partial \mathrm{U_k}}{\partial \mathrm{y_i}} = -\frac{\partial}{\partial \mathrm{y_i}} \sum_\mathrm{k} \mathrm{U_k} = -\frac{\partial \mathrm{U}}{\partial \mathrm{y_i}}$$

$$\frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial \mathrm{T}}{\partial \dot{\mathrm{y}}_\mathrm{i}} - \frac{\partial \mathrm{T}}{\partial \mathrm{y_i}} = -\frac{\partial \mathrm{U}}{\partial \mathrm{y_i}}$$

$$\frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial \mathrm{T}}{\partial \dot{\mathrm{y}}_\mathrm{i}} - \frac{\partial}{\partial \mathrm{y_i}} (\mathrm{T} - \mathrm{U}) = 0$$

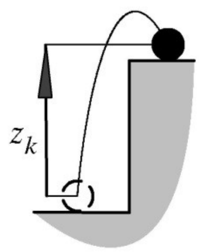
$\mathrm{L} := \mathrm{T} - \mathrm{U}$ Lagrange-Funktion

Bewegungsgleichungen
(Lagrange'sche Gleichungen zweiter Art für konservative Systeme)

$$\frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial \mathrm{L}}{\partial \dot{\mathrm{y}}_\mathrm{i}} - \frac{\partial \mathrm{L}}{\partial \mathrm{y_i}} = 0\,, \qquad \mathrm{i} = 1(1)\mathrm{f}$$

Allgemeines Vorgehen zum Aufstellen der Bewegungsgleichungen für konservative Mehrkörpersysteme

1) Beschreiben der Kinematik mit verallgemeinerten Koordinaten $\mathrm{y_1}$ ... $\mathrm{y_f}$
2) Kinetische Energie des Gesamtsystems $\mathrm{T} = \sum_\mathrm{k} \mathrm{T_k}$
   - □ Massenpunkt: $\mathrm{T_k} = \frac{1}{2} \mathrm{m_k} \mathrm{v_k^2} = \frac{1}{2} \mathrm{m_k} \mathbf{v}_\mathrm{k}^\mathrm{T} \cdot \mathbf{v}_\mathrm{k}$
   - □ Starrkörper: $\mathrm{T_k} = \frac{1}{2} \mathrm{m_k} \mathbf{v}_\mathrm{Ck}^\mathrm{T} \cdot \mathbf{v}_\mathrm{Ck} + \frac{1}{2} \boldsymbol{\omega}_\mathrm{k}^\mathrm{T} \cdot \mathbf{I}_\mathrm{Ck} \cdot \boldsymbol{\omega}_\mathrm{k}$
3) Potentielle Energie des Gesamtsystems $\mathrm{U} = \sum_\mathrm{k} \mathrm{U_k}$
   - □ Feder $\mathrm{U_k} = \frac{1}{2} \mathrm{c_k} \mathrm{s_k^2}$
   - □ Gewichtskraft $\mathrm{U_k} = \mathrm{m_k}\, \mathrm{g}\, \mathrm{z_k}$



4) Lagrange-Funktion $\mathrm{L} = \mathrm{T} - \mathrm{U}$
5) Differentiation $\dfrac{\partial \mathrm{L}}{\partial \mathrm{y_i}}$, $\dfrac{\partial \mathrm{L}}{\partial \dot{\mathrm{y}}_\mathrm{i}}$, $\dfrac{\mathrm{d}}{\mathrm{dt}}\left(\dfrac{\partial \mathrm{L}}{\partial \dot{\mathrm{y}}_\mathrm{i}}\right)$
6) **Bewegungsgleichungen** $\dfrac{\mathrm{d}}{\mathrm{dt}} \dfrac{\partial \mathrm{L}}{\partial \dot{\mathrm{y}}_\mathrm{i}} - \dfrac{\partial \mathrm{L}}{\partial \mathrm{y_i}} = 0\,, \quad \mathrm{i} = 1(1)\mathrm{f}$