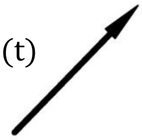
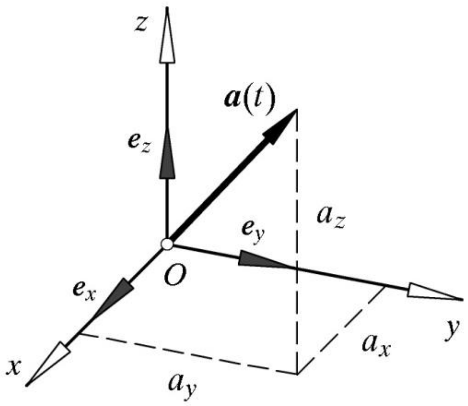
Koordinatentransformation

Koordinatendarstellung eines Vektors

Vektor $\mathbf{a}(\mathrm{t})$



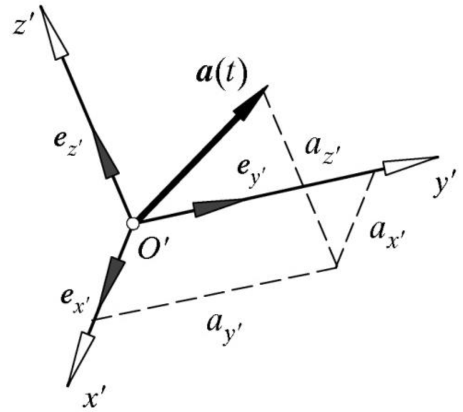
Koordinatensystem $\mathrm{K}\ \{\mathrm{O}, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$



$$\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z$$

$$\mathbf{a}_K = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

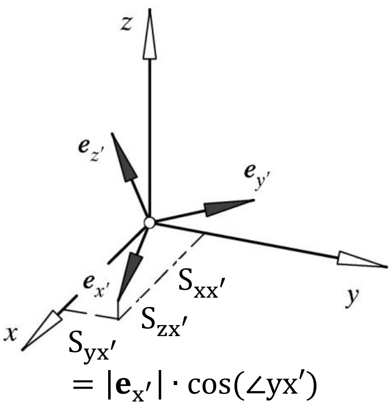
Koordinatensystem $\mathrm{K}'\ \{\mathrm{O}', \mathbf{e}_{x'}, \mathbf{e}_{y'}, \mathbf{e}_{z'}\}$



$$\mathbf{a} = a_{x'}\mathbf{e}_{x'} + a_{y'}\mathbf{e}_{y'} + a_{z'}\mathbf{e}_{z'}$$

$$\mathbf{a}_{K'} = \begin{bmatrix} a_{x'} \\ a_{y'} \\ a_{z'} \end{bmatrix}$$

Koordinatentransformation



$$\mathbf{e}_{x'} = S_{xx'}\mathbf{e}_x + S_{yx'}\mathbf{e}_y + S_{zx'}\mathbf{e}_z \qquad \mathbf{e}_{x'K} = \begin{bmatrix} S_{xx'} \\ S_{yx'} \\ S_{zx'} \end{bmatrix} = \begin{bmatrix} \cos(\angle xx') \\ \cos(\angle yx') \\ \cos(\angle zx') \end{bmatrix}$$

$$\mathbf{e}_{y'} = S_{xy'}\mathbf{e}_x + S_{yy'}\mathbf{e}_y + S_{zy'}\mathbf{e}_z \qquad \mathbf{e}_{y'K} = \begin{bmatrix} S_{xy'} \\ S_{yy'} \\ S_{zy'} \end{bmatrix} = \begin{bmatrix} \cos(\angle xy') \\ \cos(\angle yy') \\ \cos(\angle zy') \end{bmatrix}$$

$$\mathbf{e}_{z'} = S_{xz'}\mathbf{e}_x + S_{yz'}\mathbf{e}_y + S_{zz'}\mathbf{e}_z \qquad \mathbf{e}_{y'K} = \begin{bmatrix} S_{xz'} \\ S_{yz'} \\ S_{zz'} \end{bmatrix} = \begin{bmatrix} \cos(\angle xz') \\ \cos(\angle yz') \\ \cos(\angle zz') \end{bmatrix}$$

$$\mathbf{a} = a_{x'}\,\mathbf{e}_{x'} + a_{y'}\,\mathbf{e}_{y'} + a_{z'}\,\mathbf{e}_{z'}$$

Darstellung in K

$$\mathbf{a}_K = a_{x'}\,\mathbf{e}_{x'K} + a_{y'}\,\mathbf{e}_{y'K} + a_{z'}\,\mathbf{e}_{z'K}$$

$$= a_{x'}\begin{bmatrix} S_{xx'} \\ S_{yx'} \\ S_{zx'} \end{bmatrix} + a_{y'}\begin{bmatrix} S_{xy'} \\ S_{yy'} \\ S_{zy'} \end{bmatrix} + a_{z'}\begin{bmatrix} S_{xz'} \\ S_{yz'} \\ S_{zz'} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}}_{\mathbf{a}_K} = \underbrace{\begin{bmatrix} S_{xx'} & S_{xy'} & S_{xz'} \\ S_{yx'} & S_{yy'} & S_{yz'} \\ S_{zx'} & S_{zy'} & S_{zz'} \end{bmatrix}}_{\mathbf{S}_{KK'}} \cdot \underbrace{\begin{bmatrix} a_{x'} \\ a_{y'} \\ a_{z'} \end{bmatrix}}_{\mathbf{a}_{K'}} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \cdot \begin{bmatrix} a_{x'} \\ a_{y'} \\ a_{z'} \end{bmatrix}$$

Transformationsmatrix $\mathrm{K}' \rightarrow \mathrm{K}$ (Drehmatrix)

Eigenschaften einer Drehmatrix: $\mathbf{S}_{KK'} = [\mathbf{e}_{x'K} \quad \mathbf{e}_{y'K} \quad \mathbf{e}_{z'K}]$

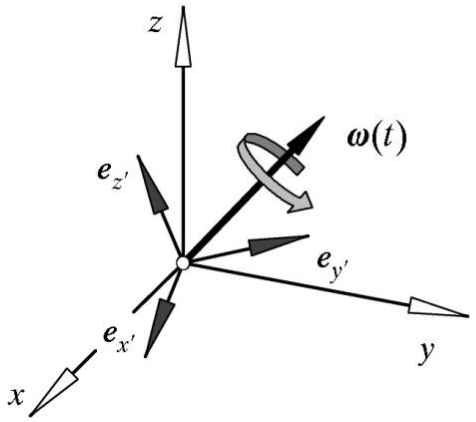
Es gilt: $\|\mathbf{e}_{x'}\| = \|\mathbf{e}_{y'}\| = \|\mathbf{e}_{z'}\| = 1$, $\quad \mathbf{e}_{x'} \cdot \mathbf{e}_{y'} = \mathbf{e}_{x'} \cdot \mathbf{e}_{z'} = \mathbf{e}_{y'} \cdot \mathbf{e}_{z'} = 0$

Daraus folgt:

$$\mathbf{S}_{KK'}^T \cdot \mathbf{S}_{KK'} = \begin{bmatrix} \mathbf{e}_{x'K} \\ \mathbf{e}_{y'K} \\ \mathbf{e}_{z'K} \end{bmatrix} \cdot [\mathbf{e}_{x'K} \quad \mathbf{e}_{y'K} \quad \mathbf{e}_{z'K}] = \begin{bmatrix} \mathbf{e}_{x'} \cdot \mathbf{e}_{x'} & \mathbf{e}_{x'} \cdot \mathbf{e}_{y'} & \mathbf{e}_{x'} \cdot \mathbf{e}_{z'} \\ \mathbf{e}_{y'} \cdot \mathbf{e}_{x'} & \mathbf{e}_{y'} \cdot \mathbf{e}_{y'} & \mathbf{e}_{y'} \cdot \mathbf{e}_{z'} \\ \mathbf{e}_{z'} \cdot \mathbf{e}_{x'} & \mathbf{e}_{z'} \cdot \mathbf{e}_{y'} & \mathbf{e}_{z'} \cdot \mathbf{e}_{z'} \end{bmatrix}_K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \mathbf{S}_{KK'}^T \cdot \mathbf{S}_{KK'} = \mathbf{S}_{KK'} \cdot \mathbf{S}_{KK'}^T = \mathbf{E}$$

Drehmatrix ist orthogonal



- ◇ $\mathbf{S}_{KK'}^{-1} = \mathbf{S}_{KK'}^T$
- ◇ $\det \mathbf{S}_{KK'} = 1$
- ◇ $\dot{\mathbf{S}}_{KK'} \cdot \mathbf{S}_{KK'}^T = \tilde{\boldsymbol{\omega}}_K = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$

$$\longrightarrow \boldsymbol{\omega}_K = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Winkelgeschwindigkeitsvektor dargestellt in K

Rücktransformation: $\mathbf{a}_{K'} = \mathbf{S}_{KK'}^T \cdot \mathbf{a}_K$ mit $\mathbf{S}_{K'K} = \mathbf{S}_{KK'}^T$