



Mathematische Grundlagen

Matrizenalgebra und Matrizenanalysis

Skalar $\mu \in \mathbb{R}$

Vektoren $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $x_i \in \mathbb{R}$

$\mathbf{y} \in \mathbb{R}^m$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$, $y_i \in \mathbb{R}$

Matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{A} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$, $A_{ij} \in \mathbb{R}$

Elementare Operationen

Operation	Schreibweise	Komponenten	Abbildungung
Addition	$\mathbf{C} = \mathbf{A} + \mathbf{B}$	$C_{ij} = A_{ij} + B_{ij}$	$\mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$
Multiplikation mit Skalar	$\mathbf{C} = \mu \mathbf{A}$	$C_{ij} = \mu A_{ij}$	$\mathbb{R} \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$
Transponieren	$\mathbf{C} = \mathbf{A}^T$	$C_{ij} = A_{ji}$	$\mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{n \times m}$
Differentiation	$\mathbf{C} = \frac{d}{dt} \mathbf{A}$	$C_{ij} = \frac{d}{dt} A_{ij}$	$\mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$
	$\mathbf{C} = \frac{\partial \mathbf{x}}{\partial \mathbf{y}}$	$C_{ij} = \frac{\partial x_i}{\partial y_j}$	$\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n \times m}$
Matrizenmultiplikation	$\mathbf{y} = \mathbf{A} \cdot \mathbf{x}$	$y_i = \sum_k A_{ik} x_k$	$\mathbb{R}^{m \times n} \times \mathbb{R}^n \rightarrow \mathbb{R}^m$
	$\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$	$C_{ij} = \sum_k A_{ik} B_{kj}$	$\mathbb{R}^{m \times n} \times \mathbb{R}^{n \times p} \rightarrow \mathbb{R}^{m \times p}$
Inneres Produkt (Skalarprodukt)	$\mu = \mathbf{x} \cdot \mathbf{z}$	$\mu = \sum_k x_k z_k$	$\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$
Äußeres Produkt (dyadisches Produkt)	$\mathbf{A} = \mathbf{x} \mathbf{y}$	$A_{ij} = x_i y_j$	$\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n \times m}$



Rechenregeln

Addition:

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

Multiplikation mit Skalar:

$$\mu(\mathbf{A} \cdot \mathbf{B}) = (\mu\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (\mu\mathbf{B})$$

$$\mu(\mathbf{A} + \mathbf{B}) = \mu\mathbf{A} + \mu\mathbf{B}$$

Transposition:

$$(\mathbf{A}^T)^T = \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$(\mu \mathbf{A})^T = \mu \mathbf{A}^T$$

$$(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T$$

Differentiation:

$$\frac{d}{dt}(\mathbf{A} + \mathbf{B}) = \frac{d}{dt}\mathbf{A} + \frac{d}{dt}\mathbf{B}$$

$$\frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) = \left(\frac{d}{dt}\mathbf{A}\right) \cdot \mathbf{B} + \mathbf{A} \cdot \left(\frac{d}{dt}\mathbf{B}\right)$$

$$\frac{d}{dt}\mathbf{x}(\mathbf{y}) = \frac{\partial \mathbf{x}}{\partial \mathbf{y}} \cdot \frac{d\mathbf{y}}{dt}$$

Matrizenmultiplikation:

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

$$\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C}$$

$$\text{aber i.A. } \mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$$

Skalarprodukt:

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$$

$$\mathbf{x} \cdot \mathbf{x} \geq 0 \quad \forall \mathbf{x}, \quad \mathbf{x} \cdot \mathbf{x} = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$$

$$\mathbf{x} \cdot \mathbf{y} = 0 \Leftrightarrow \mathbf{x}, \mathbf{y} \text{ orthogonal}$$

Quadratische Matrizen

Einheitsmatrix

$$\mathbf{E} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

Diagonalmatrix

$$\mathbf{D} = \text{diag}\{d_i\} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

Inverse Matrix

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \text{adj} \mathbf{A}$$

$$\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{E}$$

$$(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$$

Symmetrische Matrix

$$\mathbf{A}^T = \mathbf{A}$$

Schiefsymmetrische Matrix $\mathbf{A} = -\mathbf{A}^T$

Zerlegung

$$\mathbf{A} = \underbrace{\frac{1}{2}(\mathbf{A} + \mathbf{A}^T)}_{\mathbf{B} = \mathbf{B}^T} + \underbrace{\frac{1}{2}(\mathbf{A} - \mathbf{A}^T)}_{\mathbf{C} = -\mathbf{C}^T}$$



Schiefsymmetrische 3×3 Matrix

$$\tilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{a}} \cdot \mathbf{b} = \mathbf{a} \times \mathbf{b}$$

$$\tilde{\mathbf{a}} \cdot \mathbf{b} = -\tilde{\mathbf{b}} \cdot \mathbf{a} = \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$\tilde{\mathbf{a}} \cdot \tilde{\mathbf{b}} = \mathbf{b}\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{E}$$

$$\widetilde{(\tilde{\mathbf{a}} \cdot \tilde{\mathbf{b}})} = \mathbf{b}\mathbf{a} - \mathbf{a}\mathbf{b}$$

↕ Rösselsprung

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Symmetrische, positiv definite Matrix:

↔ Hauptabschnittsdeterminanten

↔ Eigenwerte

$$\mathbf{x} \cdot \mathbf{A} \cdot \mathbf{x} > 0 \quad \forall \mathbf{x} \neq \mathbf{0}$$

$$H_\alpha > 0, \quad \alpha = 1(1)n$$

$$\lambda_\alpha > 0, \quad \alpha = 1(1)n$$

Symmetrische, positiv semidefinite Matrix:

↔ Hauptabschnittsdeterminanten

↔ Eigenwerte

$$\mathbf{x} \cdot \mathbf{A} \cdot \mathbf{x} \geq 0 \quad \forall \mathbf{x}$$

$$H_\alpha \geq 0, \quad \alpha = 1(1)n$$

$$\lambda_\alpha \geq 0, \quad \alpha = 1(1)n$$

Orthogonale Matrix

$$\mathbf{A}^{-1} = \mathbf{A}^T, \quad \mathbf{A} \cdot \mathbf{A}^T = \mathbf{A}^T \cdot \mathbf{A} = \mathbf{E}$$

Determinante

$$\det \mathbf{A} = \sum_{i=1}^n A_{ik} B_{ik} = \sum_{k=1}^n A_{ik} B_{ik}$$

Adjungierte Matrix

$$\text{adj } \mathbf{A} = \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nn} \end{bmatrix}^T$$

Adjunkte eines Elementes einer Matrix

$$B_{ik} = (-1)^{i+k} \det \begin{bmatrix} A_{11} & \cdots & A_{1,k-1} & A_{1,k+1} & \cdots & A_{1n} \\ \vdots & \ddots & & & & \vdots \\ A_{i-1,1} & & & & & A_{i-1,n} \\ A_{i+1,1} & & & & & A_{i+1,n} \\ \vdots & & & & \ddots & \vdots \\ A_{n1} & \cdots & A_{n,k-1} & A_{n,k+1} & \cdots & A_{nn} \end{bmatrix}$$