3 Tools from Matrix Theory

To be able to acquire the basics of model reduction, some basics from the field of matrix theory are useful. These include various norms, matrix decompositions, and system representations. A brief overview about these topics is given in the following based on [1].

3.1 Norms

In order to be able to discuss approximation problems, it is necessary to measure the sizes of different objects. For this purpose different norms serve as comparison criterion.

Definition 3.1 *Let $\mathcal{X}$ be a real or complex vector space. A norm n is a function*

$$n : \mathcal{X} \to \mathbb{R}$$

which satisfies

i) $n(\boldsymbol{x}) \geq 0 \ \forall \boldsymbol{x} \in \mathcal{X}$ *and* $n(\boldsymbol{x}) = \boldsymbol{0} \Leftrightarrow \boldsymbol{x} = \boldsymbol{0}$ *(strictly positive definite)*

ii) $n(\lambda \boldsymbol{x}) = |\lambda| n(\boldsymbol{x}) \ \forall \boldsymbol{x} \in \mathcal{X}, \ \forall \lambda \in \mathbb{R}$ resp. $\forall \lambda \in \mathbb{C}$ *(positive homogene)*

iii) $n(\boldsymbol{x} + \boldsymbol{y}) \leq n(\boldsymbol{x}) + n(\boldsymbol{y})$ *(triangle inequality)*

In the following we will write $||\boldsymbol{x}||$ instead of $n(x)$, whereas $||\boldsymbol{x}||$ can be interpreted as the 'length' of $\boldsymbol{x}$.

Example 3.1 *Well-known examples*

1. $\mathcal{X} = \mathbb{R}^n, \ ||\boldsymbol{x}|| := \sqrt{x_1^2 + ... + x_n^2}$ *(Euclidean Norm)*
2. $\mathcal{X} = C([a, b]) =$ all continous functions $f : [a, b] \to \mathbb{R}_{||f||_\infty}, \ ||f(t)||_\infty := \max\limits_{t \in [a,b]} |f(t)|$

Below some of the most common norms are introduced

Definition 3.2 (Hölder- or p-Norm) *Let $\mathcal{X} = \mathbb{C}^n$ and $p \in [1, \infty]$*

$$||\boldsymbol{x}||_p := \begin{cases} (\sum_{i=1}^n |x_i|^p)^{(\frac{1}{p})} & \textit{if } p \in [1, \infty) \\ \max\limits_{1 \leq i \leq n} |x_i| & \textit{if } p = \infty \end{cases} \tag{3.1}$$

- $p = 2 :$ *Euclidean norm*
- $p = \infty :$ *maximum norm*

Definition 3.3 (induced matrix norm) *Let $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ and $p, q \in [1, \infty]$*

$$||\boldsymbol{A}||_{p,q} := \sup_{0 \neq \boldsymbol{x} \in \mathbb{C}^n} \frac{||\boldsymbol{A}\boldsymbol{x}||q}{||\boldsymbol{x}||_p}$$

is the (p, q) induced matrix norm of $\boldsymbol{A}$.

Motivation: The vector $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ is at most $||\boldsymbol{A}||_{p,q}$ 'longer' than $\boldsymbol{x}$.

$$||\boldsymbol{A}\boldsymbol{x}||_q \leq ||\boldsymbol{A}||_{p,q}||\boldsymbol{x}||_p.$$

Notation for simplification $||\boldsymbol{A}||_p := ||\boldsymbol{A}||_{p,p}$

Important norms Let $\boldsymbol{A} \in \mathbb{C}^{m\times n}$

- $||\boldsymbol{A}||_{1,1} = \max\limits_{1\leq j\leq n} \sum_{i=1}^{m} |A_{i,j}|$ column-sum norm
- $||\boldsymbol{A}||_{\infty,\infty} = \max\limits_{1\leq i\leq m} \sum_{i=1}^{n} |A_{i,j}|$ row-sum norm
- $||\boldsymbol{A}||_{2,2} = \sqrt{\lambda_{\max}(\boldsymbol{A}\boldsymbol{A}^H)} = \sigma_1$ spectral norm [1]
- $||\boldsymbol{A}||_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |\boldsymbol{A}_{ij}^2} = \sqrt{\sum_{i=1}^{r} \sigma_i^2(\boldsymbol{A})},$ with $r = \mathrm{rank}(\boldsymbol{A})$ Frobenius norm

[1] $\lambda_{\max}$ is the highest absolute eigenvalue and $\boldsymbol{A}^H := \boldsymbol{A}^{-T}$ the conjugate transpose of $\boldsymbol{A}$

3.2 Singular Value Decomposition

The singular value decomposition is one of the most powerful tools to decompose a matrix. It is closely related to the Eigenvalue decomposition, which is why a quick introduction about eigenvalues and eigenvectors is given below.

Repetition 3.1 (eigenvalue, eigenvector) *Let $\boldsymbol{A}_{ij} \in \mathbb{C}^{n \times n}$ be a quadratic matrix. $\lambda \in \mathbb{C}$ is called* ***eigenvalue*** *of $\boldsymbol{A}$ if there exists a vector $\boldsymbol{v} \in \mathbb{C}^n \neq 0$ with*

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$$

$\boldsymbol{v}$ is called ***eigenvector*** *to λ (one of many!)*

Remarks:

- $\mathrm{eig}_\lambda(\boldsymbol{A}) := \mathrm{Kern}(\boldsymbol{A} - \lambda \boldsymbol{I}_n) = \boldsymbol{v} \in \mathbb{C}^n 0\} :\ (\boldsymbol{A} - \lambda \boldsymbol{I}_n)\boldsymbol{v} = \boldsymbol{0}$ is called **eigenspace** of $\boldsymbol{A}$ to λ and consists of all eigenvectors to λ.
- $\mathrm{spec}(\boldsymbol{A}) : \{\lambda \in \mathbb{C} : \det(\boldsymbol{A} - \lambda \boldsymbol{I}_n) = 0\}$ is called **spectrum** of $\boldsymbol{A}$ and consists of all eigenvalues of $\boldsymbol{A}$
- If $\boldsymbol{A}$ is **Hermetian** ($\boldsymbol{A} = \boldsymbol{A}^H$), then all eigenvalues are real

Theorem 3.1 (Eigenvalue Decomposition (EVD)) *Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ with mutually distinct eigenvalues ($\forall i, j :\ \lambda_i \neq \lambda_j$), then there exists an invertible matrix $\boldsymbol{U} \in \mathbb{C}^{n \times n}$ so that*

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}$$

where $\boldsymbol{\Lambda} = diag(\lambda_1, ..., \lambda_n)$ the matrix with all n eigenvalues on its diagonal.

Remarks:

- An EVD is also possible with weaker assumptions
- $\boldsymbol{\Lambda}$ is unique if you fix the order of $(\lambda_1 ... \lambda_n)$, but $\boldsymbol{U}$ is not unique

Theorem 3.2 (Singular Value Decomposition (SVD)) *Let $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ with rank r. Then there exist unitary matrices $\boldsymbol{U} = [\boldsymbol{u}_1 \ldots \boldsymbol{u}_m] \in \mathbb{C}^{m \times m}$, $\boldsymbol{V} = [\boldsymbol{v}_1 \ldots \boldsymbol{v}_n] \in \mathbb{C}^{n \times n}$ with $\boldsymbol{U}^H\boldsymbol{U} = \boldsymbol{I}_m$, $\boldsymbol{V}^H\boldsymbol{V} = \boldsymbol{I}_n$ and a diagonal matrix*

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & & & & & \boldsymbol{0} \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ \boldsymbol{0} & & & & & 0 \end{bmatrix} \left.\vphantom{\begin{matrix} 0 \\ \ddots \\ 0 \end{matrix}}\right\} m-r \in \mathbb{R}^{m \times n}, \quad \text{Example: } \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 5}$$

$$\underbrace{\qquad\qquad}_{n-r}$$

with ***singular values*** *$\sigma_1 \geq ... \geq \sigma_r \geq 0$ so that*

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H. \tag{3.2}$$

The columns of $\boldsymbol{U}$ and $\boldsymbol{V}$ are called ***left respective right singular vectors.***

Theorem 3.3 (SVD and EVD) *Let* $\boldsymbol{A} \in \mathbb{C}^{m\times n}$

i) The singular values of $\boldsymbol{A}$ *are the square roots of the non-zero eigenvalues of* $\boldsymbol{A}^H\boldsymbol{A} \in \mathbb{C}^{n\times n}$

$$\{\sigma_i(\boldsymbol{A})\}_{i=1}^r = \{\sqrt{\lambda_i(\boldsymbol{A}^H\boldsymbol{A})}\}_{i=1}^n \setminus \{0\}.$$

ii) The columns of $\boldsymbol{U} = [\boldsymbol{u}_1 ... \boldsymbol{u}_m]$ *resp.* $\boldsymbol{V} = [\boldsymbol{v}_1 ... \boldsymbol{v}_n]$ *of the SVD (3.2) are the eigenvectors of* $\boldsymbol{A}\boldsymbol{A}^H$ *resp.* $\boldsymbol{A}^H\boldsymbol{A}$*:*

$$\boldsymbol{A}\boldsymbol{A}^H\boldsymbol{u}_i = \sigma_i^2\boldsymbol{u}_i,\ \ \boldsymbol{A}^H\boldsymbol{A}\boldsymbol{v}_i = \sigma_i^2\boldsymbol{v}_i \ \textit{for}\ 1 \leq i \leq r$$

iii) Let $\boldsymbol{A}$ *be unitary* ($\boldsymbol{A} = \boldsymbol{A}^H$) *and positive semi-definite* ($\forall i : \lambda_i \geq 0$)*, then the EVD with ordered eigenvalues* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ *is an SVD.*

Theorem 3.4 (SVD as best low-rank approximation) *Let* $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ *with rank* r *and the singular value decomposition* $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H$ *with*

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & & & & & & & \mathbf{0} \\ & \ddots & & & & & & \\ & & \sigma_k & & & & & \\ & & & \ddots & & & & \\ & & & & \sigma_r & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ \mathbf{0} & & & & & & & 0 \end{bmatrix}.$$

Define $\hat{\boldsymbol{A}}_k := \boldsymbol{U}\hat{\boldsymbol{\Sigma}}_k\boldsymbol{V}^H$ *for arbitrary* $k \in \{1, ..., r\}$ *with*

$$\hat{\boldsymbol{\Sigma}}_k = \begin{bmatrix} \sigma_1 & & & & & \mathbf{0} \\ & \ddots & & & & \\ & & \sigma_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ \mathbf{0} & & & & & 0 \end{bmatrix}.$$

Then $\hat{\boldsymbol{A}}_k$ *is the optimal solution to the problem of approximating* $\boldsymbol{A}$ *by a matrix of lower rank with respect to the spectral resp. Frobenius norm, i.e.*

i) $\min\limits_{\substack{\boldsymbol{B}\in\mathbb{C}^{m\times n} \\ rank(\boldsymbol{B})\leq k}} ||\boldsymbol{A} - \boldsymbol{B}||_2 = ||\boldsymbol{A} - \hat{\boldsymbol{A}}_k||_2 = \sigma_{k+1}$

ii) $\min\limits_{\substack{\boldsymbol{B}\in\mathbb{C}^{m\times n} \\ rank(\boldsymbol{B})\leq k}} ||\boldsymbol{A} - \boldsymbol{B}||_F = ||\boldsymbol{A} - \hat{\boldsymbol{A}}_k||_F = \sqrt{\sum_{i=k+1}^r \sigma_i^2}$

3.3 Linear Dynamical Systems

In this section some basic results about linear dynamical systems are presented including different representation schemes as well as stability properties. For this it is assumed that external variables are partitioned into **input variables $\boldsymbol{u}$** and **output variables $\boldsymbol{y}$**. When only the relation

$$\boldsymbol{y} = \boldsymbol{h} * \boldsymbol{u} \tag{3.3}$$

between $\boldsymbol{u}$ and $\boldsymbol{y}$ is known one speaks of the external description of the system. If in addition the state $\boldsymbol{x}$ is defined the internal description

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \tag{3.4}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \tag{3.5}$$

of the system can be given. The matrices $\boldsymbol{A}$, $\boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{D}$ are linear constant operators.

3.3.1 External Description

Let u be the input function of interest $\boldsymbol{u}: \mathbb{R} \to \mathbb{U} \subset \mathbb{R}^p$ which maps t onto $\boldsymbol{u}(t)$ and y the output function of interest $\boldsymbol{y}: \mathbb{R} \to \mathbb{Y} \subset \mathbb{R}^r$ mapping t onto $\boldsymbol{y}(t)$. Assume there exists a linear operator S that maps the input space $\mathbb{U}$ to the output space $\mathbb{Y}$. Then the continuous-time linear system is characterized by

$$S: \; \boldsymbol{u} \to \boldsymbol{y}(\boldsymbol{u}), \;\; \boldsymbol{y}(t) = \int_{-\infty}^{\infty} \underset{\substack{\uparrow \\ \text{kernel of the system}}}{\boldsymbol{h}(t,\tau)} \boldsymbol{u}(\tau) d\tau.$$

If the system fulfills $h(t,\tau) = 0 \; \forall \tau > t$ it is **causal**. Furthermore, if $h(t,\tau) = h(t-\tau) \; \forall t, \tau$ is satisfied in addition the system is **time-invariant** and S becomes a convolutional operator, i.e.

$$S: \; \boldsymbol{u} \to \boldsymbol{y}(\boldsymbol{u}) = S(\boldsymbol{u}) = \boldsymbol{h} * \boldsymbol{u}, \text{where } (\boldsymbol{h} * \boldsymbol{u})(t) = \int_{-\infty}^{t} \boldsymbol{h}(t-\tau)\boldsymbol{u}(\tau) d\tau.$$

For the system dynamics of a causal and time-invariant system it can be distinguished between instantaneous and purely dynamic action, i.e. the output can be build up from two terms

$$\boldsymbol{y}(t) = \boldsymbol{h}_0 \boldsymbol{u}(t) + \int_{-\infty}^{t} \boldsymbol{h}_a(t-\tau)\boldsymbol{u}(\tau) d\tau. \tag{3.6}$$

In this context $\boldsymbol{h}_0$ and $\boldsymbol{h}_a$ are smooth kernels implying that $\hbar$ can be expressed as $\boldsymbol{h}(t) = \boldsymbol{h}_0 \delta(t) + \boldsymbol{h}_a(t) \; \forall t$ with δ being the Dirac delta function. Thus, $\boldsymbol{h}$ is the response of the system to the impulse δ.
In order to calculate the input output behavior of a system it can be advantageous to do so in the frequency domain instead of the time domain. One possibility to conduct such a transformation is via Laplace transformation.

Definition 3.4 (Laplace transform (L-trafo)) *Let $f: \mathbb{R}_0^+ \to \mathbb{C}^{n \times m}$ be totally integrable, then the Laplace transform $\mathcal{L}(f): \mathbb{C} \to (\mathbb{C} \cup \{\infty\})^{n \times m}$ is defined by*

$$\mathcal{L}(f)(s) := \int_0^{\infty} e^{-st} f(t) dt, \;\; s \in \mathbb{C}. \tag{3.7}$$

Remarks

i) For vectors/matrix valued functions of f the definition is component wisie .

ii) Similar to a Fourier-transformation the L-trafo is a linear operator which maps function $f(t),\ t \in \mathbb{R}^+$ from the time domain into the **frequency domain** $F(s) := \mathcal{L}(f(t)),\ s \in \mathbb{C}$.

iii) For the inversion of $F(s) := \mathcal{L}(f(t))$ only a small subset/region of convergence of $F(s)$ in $\mathbb{C}$ is necessary. If $F(s)$ is defined on $\gamma + iT,\ T \in \mathbb{R}$ and locally integrable then $f(t) = \mathcal{L}^-1(F(s)(t)) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma-iT}^{\gamma+iT} estF(s)ds,\ \gamma \in \mathbb{R}$ what is known as Bromwhich integral.

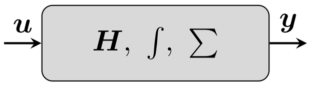
With the help of the Laplace transform of the impulse response

$$\boldsymbol{H}(s) = (\mathcal{L}(h))(s),\ s \in \mathbb{C}$$

the input-output mapping

$$\mathcal{L}(\boldsymbol{y}) = \mathcal{L}(\boldsymbol{h} * \boldsymbol{u}) \Rightarrow \boldsymbol{Y}(s) = \boldsymbol{H}(s)\boldsymbol{U}(s) \tag{3.8}$$

can be determined.



3.3.2 Internal Description

In contrast to the external description where the system is broken down to its input and output behavior a system can be described by its internal description, which in addition uses the states $\boldsymbol{x}$.

Definition 3.5 (continuous-time linear dynamical system) *Let $\boldsymbol{A}(t) \in \mathbb{R}^{N\times N},\ \boldsymbol{B}(t) \in \mathbb{R}^{N\times p}$, $\boldsymbol{C}(t) \in \mathbb{R}^{r\times N},\ \boldsymbol{D}(t) \in \mathbb{R}^{r\times p}$ for $t \in [0,\infty),\ N,p,r \in \mathbb{N}$. Then*

$$\boldsymbol{\Sigma} = \begin{cases} \frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t) \\ \boldsymbol{y} = \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t) \end{cases} \tag{3.9}$$

*is a **linear time variant (LTV) system** with input $\mathbf{u}$ state $\boldsymbol{x}$ and output $\boldsymbol{y}$. If $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ are time invariant it is a **linear time invariant (LTI) system**. The solution of the LTI is a function given by*

$$\boldsymbol{\phi}(t,\boldsymbol{u},t_0,\boldsymbol{x}_0) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_0^{t_0} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau,\ \forall t > t_0$$

and the output is

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{\phi}(t,\boldsymbol{u},t_0,\boldsymbol{x}_0) + \boldsymbol{D}\boldsymbol{u}(t).$$

By applying the Laplace transformation to (3.9) one obtains

$$s\boldsymbol{X} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}\boldsymbol{U} \Leftrightarrow \boldsymbol{X}(s\boldsymbol{I} - \boldsymbol{A}) = \boldsymbol{B}\boldsymbol{U} \Leftrightarrow \boldsymbol{X} = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U} \tag{3.10}$$

$$\boldsymbol{Y} = \boldsymbol{C}\boldsymbol{X} + \boldsymbol{D}\boldsymbol{U}, \tag{3.11}$$

where $\boldsymbol{X}$, $\boldsymbol{U}$, and $\boldsymbol{Y}$ are the Laplace transformed representations of $\boldsymbol{x}$, $\boldsymbol{u}$, or rather $\boldsymbol{y}$. Substituting (3.10) into (3.11) yields

$$\boldsymbol{Y} = (\boldsymbol{D} + \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B})\boldsymbol{U} \tag{3.12}$$

from which the transfer function

$$\boldsymbol{H}(s) = \boldsymbol{D} + \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}.$$

can easily be read off.

Transformation of the state variables Sometimes it can be advantageous to describe the system in another set of coordinates than the original one. In order to obtain transformed state variables

$$\tilde{\boldsymbol{x}} = \boldsymbol{T}\boldsymbol{x},\ det(\boldsymbol{T}) \neq 0$$

the state transformation $\boldsymbol{T}$ can be used. Applying it to (3.9) yields the transformed system

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}} = \boldsymbol{T}\dot{\boldsymbol{x}} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{x} + \boldsymbol{T}\boldsymbol{B}\boldsymbol{u} &= \overbrace{\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}}^{\tilde{\boldsymbol{A}}}\tilde{\boldsymbol{x}} + \overbrace{\boldsymbol{T}\boldsymbol{B}}^{\tilde{\boldsymbol{B}}}\boldsymbol{u} \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} &= \underbrace{\boldsymbol{C}\boldsymbol{T}^{-1}}_{\tilde{\boldsymbol{C}}}\tilde{\boldsymbol{x}} + \underbrace{\boldsymbol{D}}_{\tilde{\boldsymbol{D}}}\boldsymbol{u} \end{aligned}. \tag{3.13}$$

Remark: The transfer function of a system, which was transformed with an equivalence transformation $\boldsymbol{T}$, is equivalent to the original one $\boldsymbol{H}(s) = \tilde{\boldsymbol{H}}(s)$.

3.3.3 Stability

A linear autonomous system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t),\ \boldsymbol{A} \in \mathbb{R}^{n\times n}$ is

- **asymptotically stable** if $Re(\lambda_j) < 0$ holds for all eigenvalues of $\boldsymbol{A}$
- **stable** if and only if $Re(\lambda_j) \leq 0$ holds for all eigenvalues of $\boldsymbol{A}$ and, in addition, all purely imaginary eigenvalues have multiplicity one
- **unstable** if $Re(\lambda_j) > 0$ holds for at least one eigenvalue of $\boldsymbol{A}$ or if one eigenvalue with $Re(\lambda_j) = 0$ and multiplicity

References

[1] Antoulas, A.: Approximation of Large-Scale Dynamical Systems. Philadelphia: SIAM, 2005.