Parts of the text are extracted from [1] and [2].

Component Mode Synthesis

To improve modal reduction, the modal eigenmodes are extended with s-called *correction modes*, see Fig. 1. Such methods are frequently called *component mode synthesis* (CMS).

These methods were introduced in the 1960s [3, 4, 5]. Due to the small computational power—especially small memory—in the 1960s, large finite element models could not be solved in one run. It was necessary to subdivide the large finite element structures into small substructures. Afterwards, the movement of the substructure is expressed by different ansatz functions (*component modes*) representing the Ritz vectors of the reduced submodels. In a next step, those components are coupled (*synthesis*) together to component models, which can be solved.

The main idea of component mode synthesis methods is treating a complex structure as a set of distinct regions or substructures [5]. In this context, geometrical compatibility along substructure boundaries has to be assured. The substructures can for example be parts of a linear FE model leading to a smaller amount of degrees of freedom per component. Therefore, they are easier to handle from a numerical point of view. The substructuring idea has recently been addressed in a number of applications, e.g. [6].

Elastic multibody systems are per se set up in a modular fashion.

CMS methods combine various types of ansatz functions (component modes) that serve different purposes such as the inter-component compatibility or the description of free vibrations to represent the movement of one component with as few ansatz functions as possible, see [7] for an overview.

Arbitrarily defined deformation vectors of a single component can be used as *component/correction modes*. In the following, several types of ansatz functions and model reduction schemes are introduced and the frequently used Craig-Bampton method is briefly explained. For the sake of brevity, quantities that are obviously time dependent will not be marked explicitly.

Condensation

Two sub-groups of the condensation method can be identified: static and dynamic condensation. *Static condensation* was first introduced in [3] and is often referred to as *Guyan method*. Both techniques rest upon partitioning the nodal displacement vector into two sets of coordinates $\begin{bmatrix}\boldsymbol{q}_\mathrm{b}^T & \boldsymbol{q}_\mathrm{i}^T\end{bmatrix} = \boldsymbol{q}_\mathrm{e}^T \cdot \boldsymbol{P}^T$ with a permutation matrix $\boldsymbol{P} \in \mathbb{R}^{N\times N}$.

The conservative equation of motion in LTI representation reads in permuted form

$$\begin{aligned}\begin{bmatrix}\boldsymbol{M}_\mathrm{bb} & \boldsymbol{M}_\mathrm{bi}\\ \boldsymbol{M}_\mathrm{ib} & \boldsymbol{M}_\mathrm{ii}\end{bmatrix}\cdot\begin{bmatrix}\ddot{\boldsymbol{q}}_\mathrm{b}\\ \ddot{\boldsymbol{q}}_\mathrm{i}\end{bmatrix} + \begin{bmatrix}\boldsymbol{K}_\mathrm{bb} & \boldsymbol{K}_\mathrm{bi}\\ \boldsymbol{K}_\mathrm{ib} & \boldsymbol{K}_\mathrm{ii}\end{bmatrix}\cdot\begin{bmatrix}\boldsymbol{q}_\mathrm{b}\\ \boldsymbol{q}_\mathrm{i}\end{bmatrix} &= \begin{bmatrix}\boldsymbol{B}_\mathrm{b}\\ \boldsymbol{B}_\mathrm{i}\end{bmatrix}\cdot\boldsymbol{u}\\ \boldsymbol{y} &= \begin{bmatrix}\boldsymbol{C}_\mathrm{b} & \boldsymbol{C}_\mathrm{i}\end{bmatrix}\cdot\begin{bmatrix}\boldsymbol{q}_\mathrm{b}\\ \boldsymbol{q}_\mathrm{i}\end{bmatrix}\end{aligned} \tag{1}$$

with permuted mass and stiffness matrices $\boldsymbol{P}^T \cdot \{\boldsymbol{M}_\mathrm{e}, \boldsymbol{K}_\mathrm{e}\} \cdot \boldsymbol{P}$. For the sake of brevity, it is now assumed that the system is already given in this representation. The transfer function, mapping the input of the system to the output is given by

$$\boldsymbol{H}(s) = \boldsymbol{C}_\mathrm{e} \cdot \left(s^2\boldsymbol{M}_\mathrm{e} + \boldsymbol{K}_\mathrm{e}\right)^{-1} \cdot \boldsymbol{B}_\mathrm{e}, \tag{2}$$

with s being the complex variable in the Laplace domain. For the static case, $\ddot{\boldsymbol{q}}_\mathrm{e} = \boldsymbol{0}$ holds and implies

$$\boldsymbol{H}(0) = \begin{bmatrix} \boldsymbol{C}_\mathrm{b} & \boldsymbol{C}_\mathrm{i} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{K}_\mathrm{bb} & \boldsymbol{K}_\mathrm{bi} \\ \boldsymbol{K}_\mathrm{ib} & \boldsymbol{K}_\mathrm{ii} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \boldsymbol{B}_\mathrm{b} \\ \boldsymbol{B}_\mathrm{i} \end{bmatrix}. \tag{3}$$

The inverse of the stiffness matrix can be expressed with the Schur complement

$$\begin{bmatrix} \boldsymbol{K}_\mathrm{bb} & \boldsymbol{K}_\mathrm{bi} \\ \boldsymbol{K}_\mathrm{ib} & \boldsymbol{K}_\mathrm{ii} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{K}_\mathrm{ii}^{-1} \cdot \boldsymbol{K}_\mathrm{ib} & \boldsymbol{I} \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{K}}_\mathrm{bb}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_\mathrm{ii}^{-1} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{K}_\mathrm{bi} \cdot \boldsymbol{K}_\mathrm{ii}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{4}$$

with $\hat{\boldsymbol{K}}_\mathrm{bb} = \boldsymbol{K}_\mathrm{bb} - \boldsymbol{K}_\mathrm{bi} \cdot \boldsymbol{K}_\mathrm{ii}^{-1} \cdot \boldsymbol{K}_\mathrm{ib}$. The transfer matrix at $s = 0$ can then be decomposed according to the sets of coordinates

$$\boldsymbol{H}(0) = \boldsymbol{C}_\mathrm{i} \cdot \boldsymbol{K}_\mathrm{ii}^{-1} \cdot \boldsymbol{B}_\mathrm{i} + \hat{\boldsymbol{C}}_\mathrm{b} \cdot \hat{\boldsymbol{K}}_\mathrm{bb}^{-1} \cdot \hat{\boldsymbol{B}}_\mathrm{b}, \tag{5}$$

with

$$\hat{\boldsymbol{B}}_\mathrm{b} = \boldsymbol{B}_\mathrm{b} - \boldsymbol{K}_\mathrm{bi} \cdot \boldsymbol{K}_\mathrm{ii}^{-1} \cdot \boldsymbol{B}_\mathrm{i}, \tag{6}$$

$$\hat{\boldsymbol{C}}_\mathrm{b} = \boldsymbol{C}_\mathrm{b} - \boldsymbol{C}_\mathrm{i} \cdot \boldsymbol{K}_\mathrm{ii}^{-1} \cdot \boldsymbol{K}_\mathrm{ib}. \tag{7}$$

The two sets of coordinates are related to boundary and internal coordinates of the system. If all b interaction degrees of freedom are accounted for in the set $\boldsymbol{q}_\mathrm{b}$, i.e. $\boldsymbol{B}_\mathrm{i} = \boldsymbol{0}$ or $\boldsymbol{C}_\mathrm{i} = \boldsymbol{0}$, the Galerkin approximation of the system with

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{K}_\mathrm{ii}^{-1} \cdot \boldsymbol{K}_\mathrm{ib} \end{bmatrix}, \quad \boldsymbol{V} \in \mathbb{R}^{N \times b} \tag{8}$$

guarantees static exactness for the reduced system meaning that the transfer function of original system and reduced system match for the static case, $\boldsymbol{H}(0) = \bar{\boldsymbol{H}}(0)$.

A vivid interpretation of the ansatz functions—stored column-wise in $\boldsymbol{V}$—is that a unit displacement is applied to a single boundary degree of freedom while all other boundary degrees of freedom are fixed and the system is internally force-free,

$$\begin{bmatrix} \boldsymbol{K}_\mathrm{bb} & \boldsymbol{K}_\mathrm{bi} \\ \boldsymbol{K}_\mathrm{ib} & \boldsymbol{K}_\mathrm{ii} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{\Psi}_\mathrm{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{0} \end{bmatrix}, \tag{9}$$

see [7]. The static condensation is used as a mode set in the Craig-Bampton method, called *constraint modes* in this context. The concept of condensation can be extended to arbitrary frequencies by invoking the mass matrix and, therefore, inertia forces, in the calculation see [8]. The same procedure as above for a shift $\hat{s} = \mathrm{i}\omega$ leads to

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{I} \\ -\left(-\omega^2 \boldsymbol{M}_\mathrm{ii} + \boldsymbol{K}_\mathrm{ii}\right)^{-1} \cdot \left(-\omega^2 \boldsymbol{M}_\mathrm{ib} + \boldsymbol{K}_\mathrm{ib}\right) \end{bmatrix}, \quad \boldsymbol{V} \in \mathbb{R}^{N \times b}. \tag{10}$$

Additional Literature

Further information regarding substructureing can be found in [9] or in the overview paper of Daniel Rixen in the Gamm Rundbrief (in German) [10].

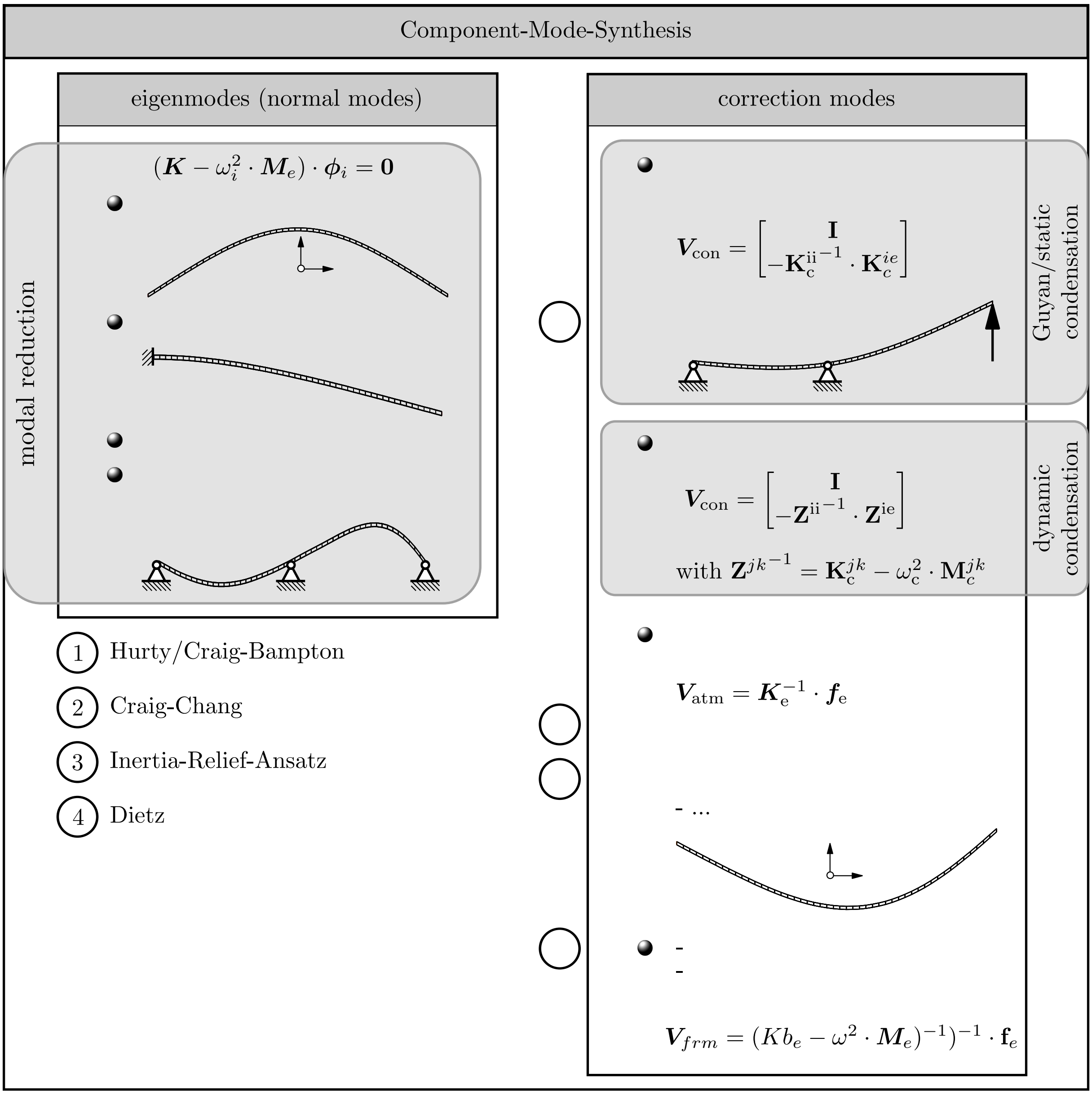

Figure 1: Component-Mode-Synthesis

Assignments for Lecture 7

1. Study and read M06 and [7].
2. Please insert the following words to the bullets and dashes in Fig. 1
   - fixed-free eigenmodes
   - (dynamic) constraints
   - ... eigenmodes
   - (dynamic) forces / frequency response modes
   - eigenmodes for complete fixed interfaces

- - ...
- static modes (forces,) attachment modes
- Inertia-Relief attachment modes
- free-free eigenmodes
- residual attachment modes
- static modes (constraints), constraint modes

3. Four frequently used combinations of eigen- plus correction modes are named in lower left part of Fig. 1. Draw four lines between eigen- and correction modes to mark which combinations are used for the respective method.

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