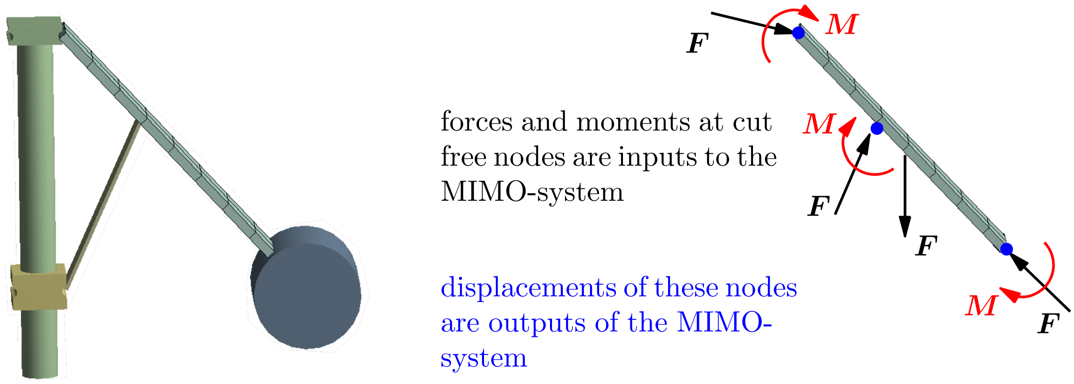

Figure 1: Cut free diagram of one elastic arm of the elastic governor

Extracted from Doctoral Thesis Jörg Fehr [1].

3.4 Description of Second order Mechanical Systems as First Order Systems

We consider a cut free diagram of an elastic body, e.g. one elastic arm of the gyroscopic governor from M3 Figure 3 is cut free in Figure 1. At those nodes where joints are attached, reaction forces apply. In addition the most important nodal displacements of the elastic body are identified. These are usually the points where joints are attached, where the body is connected to its surrounding or where measurements need to be taken. If all the reaction and applied forces acting on the elastic body are considered as input forces $\boldsymbol{B}_e \cdot \boldsymbol{u}(t)$ and the most important displacements of the elastic body as outputs $\boldsymbol{y}(t) = \boldsymbol{C}_e \cdot \boldsymbol{q}(t)$, where $\boldsymbol{B}_e \in \mathbb{R}^{N\times p}$ and $\boldsymbol{C}_e \in \mathbb{R}^{r\times N}$, then the elastic part of the body can be considered as a linear time-invariant second order MIMO system

$$\begin{aligned} \boldsymbol{M}_e \cdot \ddot{\boldsymbol{q}}(t) + \boldsymbol{D}_e \cdot \dot{\boldsymbol{q}}(t) + \boldsymbol{K}_e \cdot \boldsymbol{q}(t) &= \boldsymbol{B}_e \cdot \boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= \boldsymbol{C}_e \cdot \boldsymbol{q}(t) \end{aligned} \tag{1}$$

with the symmetric positive definite mass matrix, and the at least symmetric positive semidefinite damping and stiffness matrices

$$\boldsymbol{M}_e = \boldsymbol{M}_e^T > 0 \qquad \boldsymbol{D}_e = \boldsymbol{D}_e^T \geq 0 \qquad \boldsymbol{K}_e = \boldsymbol{K}_e^T \geq 0. \tag{2}$$

Using the Laplace transformation, e.g. explained in [2], the second order MIMO system with zero initial condition $\boldsymbol{q}(0) = \boldsymbol{0}$ is written in the complex s-domain as:

$$\begin{aligned} \left(s^2\boldsymbol{M}_e + s\boldsymbol{D}_e + \boldsymbol{K}_e\right) \cdot \boldsymbol{Q}(s) &= \boldsymbol{B}_e \cdot \boldsymbol{U}(s), \\ \boldsymbol{Y}(s) &= \boldsymbol{C}_e \cdot \boldsymbol{Q}(s), \end{aligned} \tag{3}$$

where $\boldsymbol{U}(s)$ and $\boldsymbol{Y}(s)$ are the Laplace transforms of the input $\boldsymbol{u}(t)$ and output $\boldsymbol{y}(t)$ vectors. The relation between the Laplace transforms of the input vector to the output vector is called transfer function matrix

$$\boldsymbol{Y}(s) = \boldsymbol{H}(s) \cdot \boldsymbol{U}(s) \quad \to \quad \boldsymbol{H}(s) = \boldsymbol{C}_e \cdot (s^2\boldsymbol{M}_e + s\boldsymbol{D}_e + \boldsymbol{K}_e)^{-1} \cdot \boldsymbol{B}_e \tag{4}$$

of the system. In control theory the complex variable s is only evaluated for the imaginary axis $s = i\omega$ and $\omega = 2\pi f$ which is called the circular frequency. The transfer function matrix of the system is then called the frequency response matrix of the system $\boldsymbol{H}(i\omega)$ is the magnification measured at output i of a system harmonically excited with a frequency ω at input j. The second order model can be written as an equivalent descriptor state-space model

$$\begin{aligned}
\underbrace{\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_e \end{bmatrix}}_{\boldsymbol{E}} \cdot \underbrace{\begin{bmatrix} \dot{\boldsymbol{q}}(t) \\ \ddot{\boldsymbol{q}}(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}(t)} &= \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{K}_e & -\boldsymbol{D}_e \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} \boldsymbol{q}(t) \\ \dot{\boldsymbol{q}}(t) \end{bmatrix}}_{\boldsymbol{x}(t)} + \underbrace{\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{B}_e \end{bmatrix}}_{\boldsymbol{B}_f} \cdot \boldsymbol{u}(t), \\
\boldsymbol{y}(t) &= \underbrace{\begin{bmatrix} \boldsymbol{C}_e & \boldsymbol{0} \end{bmatrix}}_{\boldsymbol{C}_f} \cdot \begin{bmatrix} \boldsymbol{q}(t) \\ \dot{\boldsymbol{q}}(t) \end{bmatrix},
\end{aligned} \tag{5}$$

where the state-space matrices $\boldsymbol{E}, \boldsymbol{A}, \boldsymbol{B}_f, \boldsymbol{C}_f$ and the state vector $\boldsymbol{x}$ are introduced. If the elastic mass matrix $\boldsymbol{M}_e$ is not singular, the second order model can be written in non-descriptor form

$$\begin{aligned}
\underbrace{\begin{bmatrix} \dot{\boldsymbol{q}}(t) \\ \ddot{\boldsymbol{q}}(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}(t)} &= \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}_e^{-1} \cdot \boldsymbol{K}_e & -\boldsymbol{M}_e^{-1} \cdot \boldsymbol{D}_e \end{bmatrix}}_{\widehat{\boldsymbol{A}}} \cdot \underbrace{\begin{bmatrix} \boldsymbol{q}(t) \\ \dot{\boldsymbol{q}}(t) \end{bmatrix}}_{\boldsymbol{x}(t)} + \underbrace{\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}_e^{-1} \cdot \boldsymbol{B}_e \end{bmatrix}}_{\widehat{\boldsymbol{B}}} \cdot \boldsymbol{u}(t), \\
\boldsymbol{y}(t) &= \underbrace{\begin{bmatrix} \boldsymbol{C}_e & \boldsymbol{0} \end{bmatrix}}_{\widehat{\boldsymbol{C}}} \cdot \begin{bmatrix} \boldsymbol{q}(t) \\ \dot{\boldsymbol{q}}(t) \end{bmatrix}.
\end{aligned} \tag{6}$$

Using the Laplace transformation the transfer matrix of the first order system

$$\boldsymbol{H}_f(s) = \boldsymbol{C}_f \cdot (s\boldsymbol{E} - \boldsymbol{A})^{-1} \cdot \boldsymbol{B}_f \tag{7}$$

is obtained. Most model reduction techniques were developed for state-space systems (6) but can be usually extended for second order models by keeping in mind that the second order system can be transformed via Eq. (5) or Eq. (6) into a first order system.

References

[1] Fehr, J.: Automated and Error-Controlled Model Reduction in Elastic Multibody Systems. Dissertation, Schriften aus dem Institut für Technische und Numerische Mechanik der Universität Stuttgart, Vol. 21. Aachen: Shaker Verlag, 2011.

[2] Lunze, J.: Regelungstechnik 1, Systemtheoretische Grundlagen, Analyse und Entwurf einschleifiger Regelungen. Berlin: Springer, 2008.