Extracted from Doctoral Thesis Jörg Fehr [1].

Model Reduction of Second Order MIMO Systems

The first order state space descriptor system M04 Eq. (5) is reduced with a Petrov-Galerkin ansatz $\boldsymbol{x}(t) \approx \boldsymbol{V}_f \cdot \bar{\boldsymbol{x}}(t)$

$$\begin{aligned} \boldsymbol{E} \cdot \boldsymbol{V}_f \cdot \dot{\bar{\boldsymbol{x}}}(t) &= \boldsymbol{A} \cdot \boldsymbol{V}_f \cdot \bar{\boldsymbol{x}}(t) + \boldsymbol{B}_f \cdot \boldsymbol{u}(t) + \boldsymbol{\varepsilon}_f(t) \\ \bar{\boldsymbol{y}}(t) &= \boldsymbol{C}_f \cdot \boldsymbol{V}_f \cdot \bar{\boldsymbol{x}}(t) \end{aligned} \tag{1}$$

and requiring the residual $\boldsymbol{\varepsilon}_f(t)$ projected along the subspace $\boldsymbol{W}_f$ onto the subspace $\boldsymbol{V}_f$ to be zero, compare e.g. [2]. The reduced system then reads

$$\begin{aligned} \underbrace{\boldsymbol{W}_f^T \cdot \boldsymbol{E} \cdot \boldsymbol{V}_f}_{\bar{\boldsymbol{E}}} \cdot \dot{\bar{\boldsymbol{x}}}(t) &= \underbrace{\boldsymbol{W}_f^T \cdot \boldsymbol{A} \cdot \boldsymbol{V}_f}_{\bar{\boldsymbol{A}}} \cdot \bar{\boldsymbol{x}}(t) + \underbrace{\boldsymbol{W}_f^T \cdot \boldsymbol{B}}_{\bar{\boldsymbol{B}}} \cdot \boldsymbol{u}(t) + \underbrace{\boldsymbol{W}_f^T \cdot \boldsymbol{\varepsilon}_f(t)}_{\boldsymbol{0}} \\ \bar{\boldsymbol{y}}(t) &= \underbrace{\boldsymbol{C}_f \cdot \boldsymbol{V}_f}_{\bar{\boldsymbol{C}}_f} \cdot \bar{\boldsymbol{x}}(t) \end{aligned} \tag{2}$$

and the transfer matrix of the first order reduced system is $\bar{\boldsymbol{H}}_f(s) = \bar{\boldsymbol{C}}_f \cdot (s\bar{\boldsymbol{E}} - \bar{\boldsymbol{A}})^{-1} \cdot \bar{\boldsymbol{B}}_f$.
The second order MIMO system M 4 Eq. (2) with the original dimension N is also reduced by a Petrov-Galerkin projection of the elastic coordinates $\boldsymbol{q}$, visualized in Figure 1, on to the subspace $\operatorname{span}(\boldsymbol{V}) \in \mathbb{R}^{N \times n}$ by $\boldsymbol{q} \approx \boldsymbol{V} \cdot \bar{\boldsymbol{q}}$

$$\begin{aligned} \boldsymbol{M}_e \cdot \boldsymbol{V} \cdot \ddot{\bar{\boldsymbol{q}}}(t) + \boldsymbol{D}_e \cdot \boldsymbol{V} \cdot \dot{\bar{\boldsymbol{q}}}(t) + \boldsymbol{K}_e \cdot \boldsymbol{V} \cdot \bar{\boldsymbol{q}}(t) &= \boldsymbol{B}_e \cdot \boldsymbol{u}(t) + \boldsymbol{\varepsilon}(t), \\ \bar{\boldsymbol{y}}(t) &= \boldsymbol{C}_e \cdot \boldsymbol{V} \cdot \bar{\boldsymbol{q}}(t) \end{aligned} \tag{3}$$

and requiring the residual $\boldsymbol{\varepsilon}(t)$ projected by the subspace $\boldsymbol{W}$ onto the subspace $\boldsymbol{V}$ to be zero. The reduced system of size n then reads

$$\begin{aligned} \bar{\boldsymbol{M}}_e \cdot \ddot{\bar{\boldsymbol{q}}}(t) + \bar{\boldsymbol{D}}_e \cdot \dot{\bar{\boldsymbol{q}}}(t) + \bar{\boldsymbol{K}}_e \cdot \bar{\boldsymbol{q}}(t) &= \bar{\boldsymbol{B}}_e \cdot \boldsymbol{u}(t), \\ \bar{\boldsymbol{y}}(t) &= \bar{\boldsymbol{C}}_e \cdot \bar{\boldsymbol{q}}(t). \end{aligned} \tag{4}$$

The projection spaces $\operatorname{span}(\boldsymbol{V})$ and $\operatorname{span}(\boldsymbol{W})$ which reduces the full elastic body M 3 Eq. (12) to its reduced form, compare Equation (5), are then the same projection spaces which reduce the second

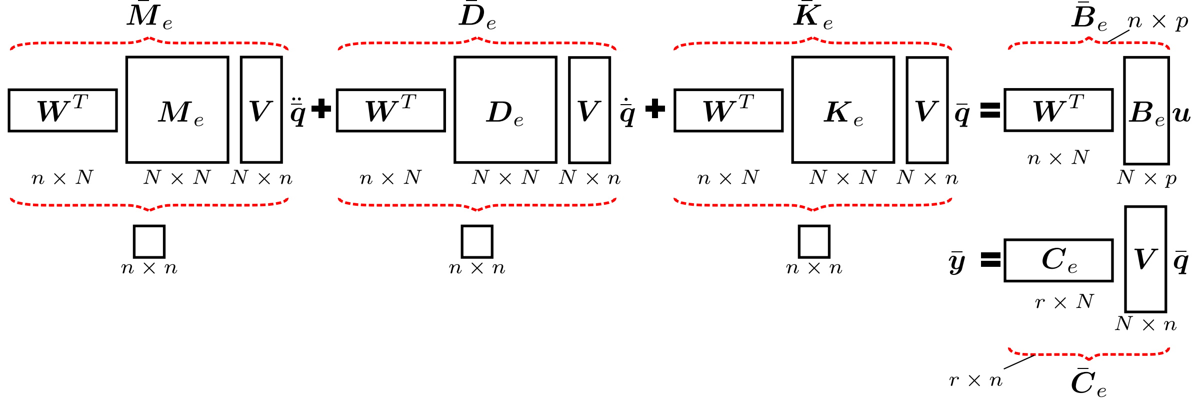

Figure 1: Model reduction of a second order MIMO system via Petrov-Galerkin projection

order MIMO-system

$$\begin{bmatrix} m\boldsymbol{I} & m\tilde{\bar{\boldsymbol{c}}}^T(\bar{\boldsymbol{q}}) & {}^v\bar{\boldsymbol{C}}_t^T \cdot \boldsymbol{V} \\ m\tilde{\bar{\boldsymbol{c}}} & \bar{\boldsymbol{J}}(\bar{\boldsymbol{q}}) & {}^v\bar{\boldsymbol{C}}_r^T(\bar{\boldsymbol{q}}) \cdot \boldsymbol{V} \\ \boldsymbol{W}^T \cdot {}^w\bar{\boldsymbol{C}}_t(\bar{\boldsymbol{q}}) & \boldsymbol{W}^T \cdot {}^w\bar{\boldsymbol{C}}_r(\bar{\boldsymbol{q}}) & \boldsymbol{W}^T \cdot \boldsymbol{M}_e \cdot \boldsymbol{V} \end{bmatrix} \cdot \begin{bmatrix} \dot{\boldsymbol{v}}_{IR} \\ \dot{\boldsymbol{\omega}}_{IR} \\ \ddot{\bar{\boldsymbol{q}}} \end{bmatrix} = -\bar{\boldsymbol{h}}_e - \bar{\boldsymbol{h}}_\omega + \bar{\boldsymbol{h}}_g + \bar{\boldsymbol{h}}_P + \bar{\boldsymbol{h}}_d \quad (5)$$

with the reduced forces $\bar{\boldsymbol{h}}_e$, $\bar{\boldsymbol{h}}_\omega$, $\bar{\boldsymbol{h}}_g$, $\bar{\boldsymbol{h}}_P$, $\bar{\boldsymbol{h}}_d$, the inertia tensor $\bar{\boldsymbol{J}}$ and the center of gravity $\bar{\boldsymbol{c}}$ of the reduced elastic body, which are calculated from the mass invariants if the left elastic projection operator now only spans the motion of the reduced elastic motion spaces $\boldsymbol{S}_{el} = \boldsymbol{T}_{BC} \cdot \boldsymbol{V}$ and $\boldsymbol{S}_{rl} = \boldsymbol{T}_{BC} \cdot \boldsymbol{W}$. In addition, it is distinguished between ${}^v\bar{\boldsymbol{C}}_t/{}^v\bar{\boldsymbol{C}}_r$ and ${}^w\bar{\boldsymbol{C}}_t/{}^w\bar{\boldsymbol{C}}_r$ due to the non-orthogonal projection.

Importance of Second Order Model Reduction

Most of the state space reduction methods were initially developed for first order systems. However, the underlying equation of motion of a flexible multibody system is always a second order differential equation. One simple reduction method for such a class of second order systems is, to utilize the conventional first order model reduction techniques [3, 4]. However, due to the embedding into the problem of double size, the condition number, i.e. the sensitivity of the eigenvalues and eigenvectors with respect to perturbations in the data matrices may increase, compare [5]. Here the attention is turned to model reduction techniques for second order systems. After [5] they are equipped with the following desirable features:

- The algorithms work directly with the original system data avoiding the problem of increased condition numbers.
- The algorithms preserve the sparsity structure inherited from the original second order system.
- They avoid the loss of physical insight of the original system.
- The efficiency and the reliability of reduction techniques are improved.

In addition, in multibody system tools like Adams, Simpack or Neweul-M^2, the reduced elastic body (5) must be described as a second order system for the simulations and the preservation of the second order structure is mandatory.

3.3.1 Error Induced by Projection

It is desirable to have an approximation error as small as possible between the original system and the reduced system. This error needs to be measured adequately. First, a general definition of the error between the original and the reduced system is given. This error system is used to visualize the error in the frequency domain.

Error Measures for Second Order System

For a second order MIMO system, the error in the time domain in the position states is defined as

$$\boldsymbol{e}(t) = \boldsymbol{q}(t) - \boldsymbol{V} \cdot \bar{\boldsymbol{q}}(t). \quad (6)$$

The Laplace transform is used to transform Equation (6) into the complex s-domain

$$\boldsymbol{E}(s) = \boldsymbol{Q}(s) - \boldsymbol{V} \cdot \bar{\boldsymbol{Q}}(s), \tag{7}$$

with zero initial condition $\boldsymbol{e}(0) = 0$, compare [6]. The position and the reduced position vector are rephrased as $\boldsymbol{Q}(s) = (s^2\boldsymbol{M}_e + s\boldsymbol{D}_e + \boldsymbol{K}_e)^{-1} \cdot \boldsymbol{B}_e \cdot \boldsymbol{U}$ and $\bar{\boldsymbol{Q}}(s) = (s^2\bar{\boldsymbol{M}}_e + s\bar{\boldsymbol{D}}_e + \bar{\boldsymbol{K}}_e)^{-1} \cdot \bar{\boldsymbol{B}}_e \cdot \boldsymbol{U}(s)$. This leads to

$$\boldsymbol{E}(s) = (s^2\boldsymbol{M}_e + s\boldsymbol{D}_e + \boldsymbol{K}_e)^{-1} \cdot \boldsymbol{B}_e \cdot \boldsymbol{U}(s) - \boldsymbol{V} \cdot \left((s^2\bar{\boldsymbol{M}}_e + s\bar{\boldsymbol{D}}_e + \bar{\boldsymbol{K}}_e)^{-1} \cdot \bar{\boldsymbol{B}}_e \cdot \boldsymbol{U}(s)\right). \tag{8}$$

In a next step the error is expressed in terms of the residuum. The Laplace transform $\boldsymbol{E}_r(s)$ of the residuum $\boldsymbol{\varepsilon}(t)$ obtained by reduction on subspace span($\boldsymbol{V}$) can be written as

$$\boldsymbol{E}_r(s) = (s^2\boldsymbol{M}_e + s\boldsymbol{D}_e + \boldsymbol{K}_e) \cdot \boldsymbol{E}(s), \tag{9}$$

and can be written with Equation (8) as:

$$\begin{aligned}\boldsymbol{E}_r(s) &= (s^2\boldsymbol{M}_e + s\boldsymbol{D}_e + \boldsymbol{K}_e) \cdot (s^2\boldsymbol{M}_e + s\boldsymbol{D}_e + \boldsymbol{K}_e)^{-1} \cdot \boldsymbol{B}_e \cdot \boldsymbol{U}(s) \\ &\quad - (s^2\boldsymbol{M}_e + s\boldsymbol{D}_e + \boldsymbol{K}_e) \cdot \boldsymbol{V} \cdot \left((s^2\bar{\boldsymbol{M}}_e + s\bar{\boldsymbol{D}}_e + \bar{\boldsymbol{K}}_e)^{-1} \cdot \bar{\boldsymbol{B}}_e \cdot \boldsymbol{U}(s)\right) \\ &= \boldsymbol{B}_e \cdot \boldsymbol{U} - (s^2\boldsymbol{M}_e + s\boldsymbol{D}_e + \boldsymbol{K}_e) \cdot \boldsymbol{V} \left((s^2\bar{\boldsymbol{M}}_e + s\bar{\boldsymbol{D}}_e + \bar{\boldsymbol{K}}_e)^{-1} \cdot \bar{\boldsymbol{B}}_e \cdot \boldsymbol{U}(s)\right).\end{aligned} \tag{10}$$

Usually only a good input to output mapping is desired, which leads to the definition of the output error for the second order system $\boldsymbol{e}_O(t) = \boldsymbol{C}_e \cdot \boldsymbol{e}(t) = \boldsymbol{y}(t) - \bar{\boldsymbol{y}}(t)$ which is defined in the complex s-domain as:

$$\begin{aligned}\boldsymbol{E}_O(s) &= \boldsymbol{Y}(s) - \bar{\boldsymbol{Y}}(s) = \boldsymbol{C}_e \cdot \boldsymbol{Q}(s) - \bar{\boldsymbol{C}}_e \cdot \bar{\boldsymbol{Q}}(s) \\ &= \boldsymbol{C}_e \cdot \left(\boldsymbol{Q}(s) - \boldsymbol{V} \cdot \bar{\boldsymbol{Q}}(s)\right) = \boldsymbol{C}_e \cdot \boldsymbol{E}(s).\end{aligned} \tag{11}$$

Error System

The output error Equation (11) which needs to be minimized for good reduction results can also be written in terms of the error system

$$\boldsymbol{H}_E(s) = \boldsymbol{H}(s) - \bar{\boldsymbol{H}}(s). \tag{12}$$

Inserting $\boldsymbol{Y}(s) = \boldsymbol{H}(s) \cdot \boldsymbol{U}(s)$ and $\bar{\boldsymbol{Y}} = \bar{\boldsymbol{H}}(s) \cdot \boldsymbol{U}(s)$ into Equation (11) the output error reads

$$\boldsymbol{E}_O(s) = \boldsymbol{H}(s) \cdot \boldsymbol{U}(s) - \bar{\boldsymbol{H}}(s) \cdot \boldsymbol{U}(s) = \boldsymbol{H}_E(s) \cdot \boldsymbol{U}(s), \tag{13}$$

where for both the reduced and the original system a zero initial condition is considered.
As explained in [4, 7] the $\mathcal{L}_p$ norms of the error system $\boldsymbol{H}_E(s)$ are natural performance metrics. The most common induced norms are the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms. After [6] all proper rational transfer functions that are analytical and bounded in the closed right half plane are part of the Hardy space $\mathcal{H}_\infty$. The $\mathcal{H}_\infty$-norm (norm corresponding to the Hardy space $\mathcal{H}_\infty$) is the induced $\mathcal{L}_2$ norm of a system and is defined as

$$\|\boldsymbol{H}\|_{\mathcal{H}_\infty} = \sup_{\boldsymbol{u}\neq\boldsymbol{0}} \frac{\|\boldsymbol{H}\cdot\boldsymbol{u}\|_{\mathcal{L}_2^r(i\mathbb{R})}}{\|\boldsymbol{u}\|_{\mathcal{L}_2^p(i\mathbb{R})}} = \sup_{\omega\in\mathbb{R}} \|\boldsymbol{H}(i\omega)\|_2. \tag{14}$$

The error is measured in the $\mathcal{H}_\infty$-norm because this norm allows an error interpretation in the frequency domain as well as in the time domain, see [4, 6]. Due to the Parseval identity the $\mathcal{H}_\infty$-norm can also be written as

$$\|\boldsymbol{H}\|_{i,2} = \|\boldsymbol{H}\|_{\mathcal{H}_\infty} = \sup_{\boldsymbol{u}\neq\boldsymbol{0}} \frac{\|\boldsymbol{y}\|_{\mathcal{L}_2^r(\mathbb{R})}}{\|\boldsymbol{u}\|_{\mathcal{L}_2^p(\mathbb{R})}}. \tag{15}$$

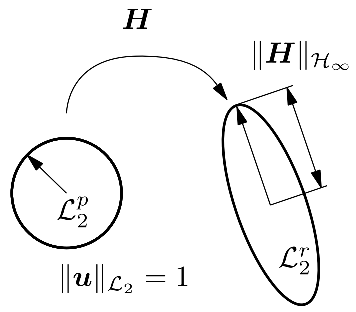

Figure 2: Input/Output Ellipsoid

Conceptually the second order system $\boldsymbol{H}$ can be thought of as mapping a unit input $\mathcal{L}_2^p$-ball to an output ellipsoid in $\mathcal{L}_2^r$ as shown in Figure 2. The worst case input/output $\mathcal{L}_2$ gain is the $\mathcal{H}_\infty$-norm of the system and is the length of the major axis of the output ellipsoid. Then the $\mathcal{H}_\infty$-norm of the error system describes the worst case error between the original and the reduced system for all inputs with a unit input gain.
Another widely used norm in model reduction is the $\mathcal{H}_2$-norm of a system. For example in [8, 9] $\mathcal{H}_2$ optimal model reduction techniques are discussed. The $\mathcal{H}_2$-norm of a system is the corresponding $\mathcal{L}_2^{p\times r}(-i\omega, i\omega)$ norm of the frequency response matrix $\boldsymbol{H}(i\omega)$, which can also be written as

$$\begin{aligned}\|\boldsymbol{H}\|_{\mathcal{H}_2} &= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}(\boldsymbol{H}^H(-i\omega)\cdot\boldsymbol{H}(i\omega))d\omega\right)^{\frac{1}{2}}\\ &= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|\boldsymbol{H}(i\omega)\|_F^2 d\omega\right)^{\frac{1}{2}}, \qquad (16)\end{aligned}$$

where

$$\|\boldsymbol{H}(i\omega)\|_F = \sqrt{\mathrm{trace}(\boldsymbol{H}(i\omega)\cdot\boldsymbol{H}^H(i\omega))} \qquad (17)$$

is the sub multiplicative Frobenius-norm of the complex matrix $\boldsymbol{H}$, see [4, 10]. In [7, 4] the time domain interpretation of the $\mathcal{H}_2$-norm is given as

$$\|\boldsymbol{H}\|_{\mathcal{H}_2} = \left(\int_{t_1}^{t_2} \mathbf{trace}[\boldsymbol{h}^H(t)\cdot\boldsymbol{h}(t)]\,dt\right)^{\frac{1}{2}}, \qquad (18)$$

where $\boldsymbol{h}(t)$ is the impulse response of the dynamical system $\boldsymbol{H}$. A small $\mathcal{H}_2$-norm of the error system $\|\boldsymbol{H}_e\|_{\mathcal{H}_2}$ means that the $\mathcal{L}_2$ norm of the difference between the output of the reduced order system and the original system if both are excited with the impulse response or, equivalently, between the transfer functions is small. There is no direct relation to time domain error bounds, however as explained in [7] a small $\mathcal{H}_2$-norm means that under unit Gaussian white noise input with unit spectral density, the power of the output is small.

Frequently, in mechanical systems a certain frequency range is of special interest. In order to evaluate a dynamical system only in a certain frequency range, the weighted $\mathcal{H}_\infty^i$-norm and $\mathcal{H}_2^i$-norm

$$\|\boldsymbol{W}_o\cdot\boldsymbol{H}\cdot\boldsymbol{W}_i\|_{\mathcal{H}_\infty^i} \qquad \|\boldsymbol{W}_o\cdot\boldsymbol{H}\cdot\boldsymbol{W}_i\|_{\mathcal{H}_2^i} \qquad (19)$$

of the dynamical system $\boldsymbol{H}$ with the still to be defined frequency weighting matrices $\boldsymbol{W}_i(s)$ and $\boldsymbol{W}_o(s)$ are introduced. If the weighting matrices are ideal band pass filters in the frequency range $[f_{min} = 2\pi\omega_{min},\ f_{max} = 2\pi\omega_{max}]$, the weighted $\mathcal{H}_\infty^i$-norm of a system is defined as

$$\|\boldsymbol{H}\|_{\mathcal{H}_\infty^i} = \sup_{f\in[f_{min},\, f_{max}]} \|\boldsymbol{H}(i\,2\pi f)\|_2 \qquad (20)$$

and the weighted $\mathcal{H}_2^i$-norm of a system is defined as

$$\|\boldsymbol{H}\|^2_{\mathcal{H}_2^i} = \frac{1}{2\pi}\int_{f_{min}}^{f_{max}} \|\boldsymbol{H}(i\,2\pi f)\|_F \, df. \tag{21}$$

To visualize the quality of the reduced order system in a certain frequency range either the spectral norm $\|\boldsymbol{H}_E(i\,2\pi f)\|_2$ or the Frobenius norm $\|\boldsymbol{H}_E(i\,2\pi f)\|_F$ of the frequency response matrix is plotted over the frequency. As an example, the error system $\boldsymbol{H}_E$ of a with four eigenmodes reduced elastic arm of the governor controller is plotted in Figure 3. The maximum of the spectral norm is the weighted $\mathcal{H}_\infty^i$-norm of the error system. The integral under the Frobenius norm is proportional to the frequency weighted $\mathcal{H}_2^i$-norm of the error system compare Equation (21). As seen in Figure 3 the absolute error is rather small due to the small norm of the original transfer matrix $\|\boldsymbol{H}\|_{F/2}$, see Figure 4. Usually higher frequencies are more damped leading to a decline of the transfer function for higher frequencies. In this example the norm of the transfer function between the original and the reduced system does not match for higher frequencies. However the absolute spectral norm and the Frobenius norm can not indicate the deviation of both systems, due to the amplitude drop caused by damping. It is appropriate to normalize the error by division with the norm of the original system $\|\boldsymbol{H}\|_{F/2}$. These errors are called relative errors and are defined as

$$\varepsilon_2^{rel}(f) = \frac{\|\boldsymbol{H}_E(i\,2\pi f)\|_2}{\|\boldsymbol{H}(i\,2\pi f)\|_2} = \frac{\|\boldsymbol{H}(i\,2\pi f) - \bar{\boldsymbol{H}}(i\,2\pi f)\|_2}{\|\boldsymbol{H}(i\,2\pi f)\|_2}, \quad f \in [f_{min},\ f_{max}], \tag{22}$$

$$\varepsilon_F^{rel}(f) = \frac{\|\boldsymbol{H}_E(i\,2\pi f)\|_F}{\|\boldsymbol{H}(i\,2\pi f)\|_F} = \frac{\|\boldsymbol{H}(i\,2\pi f) - \bar{\boldsymbol{H}}(i\,2\pi f)\|_F}{\|\boldsymbol{H}(i\,2\pi f)\|_F}, \quad f \in [f_{min},\ f_{max}]. \tag{23}$$

In Figure 3 the relative errors are also plotted in the spectral and in the Frobenius norm. Both errors show the same behavior and both errors can detect the deviation between original and reduced system at a higher frequency range. As already seen in the example there is no big difference between the spectral and the Frobenius norm because the following relation holds between the spectral and the Frobenius norm of a matrix $\boldsymbol{A}$

$$\|\boldsymbol{A}\|_F \le \sqrt{\min(\mathrm{n},\mathrm{m})}\,\|\boldsymbol{A}\|_2. \tag{24}$$

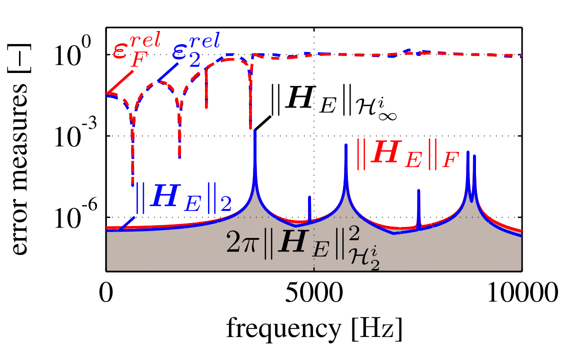

Figure 3: Different error measures between the original and the with 4 eigenmodes reduced elastic Governor arm

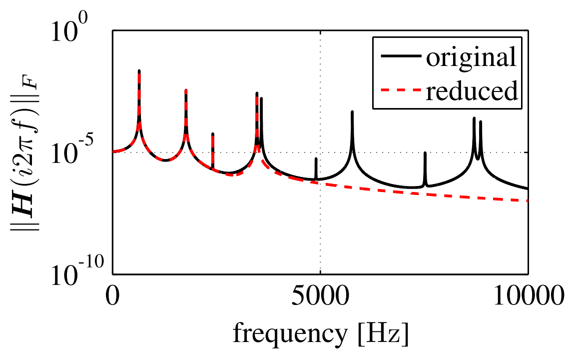

Figure 4: Transfer function of the original and the reduced elastic Governor arm

References

[1] Fehr, J.: Automated and Error-Controlled Model Reduction in Elastic Multibody Systems. Dissertation, Schriften aus dem Institut für Technische und Numerische Mechanik der Universität Stuttgart, Vol. 21. Aachen: Shaker Verlag, 2011.

[2] Eid, R.: Time Domain Model Reduction by Moment Matching. Dissertation, Technische Universität München. München: Verlag Dr. Hut, 2009.

[3] Chahlaoui, Y.; Lemonnier, D.; Vandendorpe, A.; Dooren, P.V.: Second-Order Balanced Truncation. Linear Algebra and its Application, Vol. 415, No. 2–3, pp. 378–384, 2006.

[4] Antoulas, A.: Approximation of Large-Scale Dynamical Systems. Philadelphia: SIAM, 2005.

[5] Chu, C.C.; Tsai, H.C.; Lai, M.H.: Structure Preserving Model-order Reductions of MIMO Second-order Systems using Arnoldi Methods. Mathematical and Computer Modelling, Vol. 51, No. 7, pp. 956–973, 2010.

[6] Mehrmann, V.; Stykel, T.: Balanced Truncation Model Reduction for Large-Scale Systems in Descriptor Form. In Benner, P.; Mehrmann, V.; Sorensen, D. (Eds.): Dimension Reduction of Large-Scale Systems, Vol. 45 of Lecture Notes in Computational Science and Engineering, pp. 83–115. Berlin: Springer, 2005.

[7] Gressick, W.; Wen, J.; Fish, J.: Order Reduction for Large Scale Finite Element Models: A System Perspective. International Journal for Multiscale Computational Engineering, Vol. 3, pp. 337–362, 2005.

[8] Bunse-Gerstner, A.; Kubalinska, D.; Vossen, G.; Wilczek, D.: $\mathcal{H}_2$-Norm Optimal Model Reduction for Large Scale Discrete Dynamical MIMO Systems. Journal of Computational and Applied Mathematics, Vol. 233, pp. 1202–1216, 2010.

[9] Gugercin, S.; Antoulas, A.C.; Beattie, C.A.: $\mathcal{H}_2$ Model Reduction for Large-Scale Linear Dynamical Systems. SIAM Journal on Matrix Analysis and Applications, Vol. 30, No. 2, pp. 609–638, 2008.

[10] Lehner, M.: Modellreduktion in elastischen Mehrkörpersystemen (in German). Dissertation, Schriften aus dem Institut für Technische und Numerische Mechanik der Universität Stuttgart, Vol. 10. Aachen: Shaker Verlag, 2007.