1 Cheat Sheet

1.1 Preliminaries

Norms ________

Vector x:

$$||\boldsymbol{x}||_p = \frac{\qquad}{\rule{10cm}{0.4pt}}$$

Matrix $\boldsymbol{A}$: :

$$||\boldsymbol{A}||_2 = \rule{6cm}{0.4pt}$$

$$||\boldsymbol{A}||_F = \frac{\qquad}{\rule{9cm}{0.4pt}}$$

$$= \frac{\qquad}{\rule{4.5cm}{0.4pt}}$$

Function $\boldsymbol{H}(s)$:

$$||\boldsymbol{H}(s)||_{H2} = \sqrt{\frac{\qquad}{\rule{9cm}{0.4pt}}}$$

$$\to \rule{7.5cm}{0.4pt}$$

$$||\boldsymbol{H}(s)||_{H\infty} = \frac{\qquad}{\rule{9.5cm}{0.4pt}}$$

$$\to \rule{5.5cm}{0.4pt}$$

Matrix Decompositions

Eigenvalue Decomposition (EVD)

$\boldsymbol{A} \in \mathbb{C}^{n \times n}$, $\boldsymbol{U} \in \mathbb{C}^{n \times n}$ invertible

$\boldsymbol{A} = \rule{2.5cm}{0.4pt}$

$\boldsymbol{\Lambda} = diag(\lambda_1, ..., \lambda_n)$ (eigenvalues)

$\updownarrow$

$\boldsymbol{U} = [\boldsymbol{u}_1, ..., \boldsymbol{u}_n]$ eigenvectors

Example EVD for matrix exponentials:

$$\boldsymbol{A}^5 = \boldsymbol{U}\boldsymbol{\Lambda}\cancel{\boldsymbol{U}^{-1}} \cdot ... \cdot \cancel{\boldsymbol{U}}\boldsymbol{\Lambda}\boldsymbol{U}^{-1} = \boldsymbol{U}\boldsymbol{\Lambda}^5\boldsymbol{U}^{-1}$$

Singular Value Decomposition (SVD)

$\boldsymbol{A} \in \mathbb{C}^{m \times n}$, $\boldsymbol{U} \in \mathbb{C}^{m \times m}$, $\boldsymbol{V} \in^{n \times n}$

$$\boldsymbol{U}^H\boldsymbol{U} = I, \boldsymbol{V}^H\boldsymbol{V} = I$$

$\boldsymbol{A} = \rule{2.5cm}{0.4pt}$

$\boldsymbol{\Sigma} = diag(\sigma_1, ..., \sigma_r, 0, ..., 0)$ (sing values)

$\updownarrow$

$\boldsymbol{U} = [\boldsymbol{u}_1, ..., \boldsymbol{u}_m], \boldsymbol{V} = [\boldsymbol{v}_1, ..., \boldsymbol{v}_n]$ (sing vectors)

$\sigma_i = \sqrt{\lambda_i(\boldsymbol{A}^H\boldsymbol{A})}$

$\boldsymbol{u}_i, \boldsymbol{v}_i$ eigenvector of $\boldsymbol{A}^H\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{A}^H$

$\boldsymbol{A}\boldsymbol{A}^H\boldsymbol{u}_i = \sigma_i^2\boldsymbol{u}_i$

$\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{v}_i = \sigma_i^2\boldsymbol{v}_i$

SVD as best low-rank approximation

$$\hat{\boldsymbol{A}}_k = \underline{\hspace{3cm}},$$

$$\boldsymbol{\Sigma}_k = \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_k & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$$

$$||\boldsymbol{A} - \hat{\boldsymbol{A}}_k||_2 = \underline{\hspace{1.5cm}}$$

$$||\boldsymbol{A} - \hat{\boldsymbol{A}}_k||_F = \underline{\hspace{3.5cm}}$$

Transformation of state variables $\boldsymbol{x} \rightarrow \boldsymbol{T}\boldsymbol{x}$

$$\tilde{\boldsymbol{x}} = \boldsymbol{T}\boldsymbol{x},\ \dot{\tilde{\boldsymbol{x}}} = \boldsymbol{T}\dot{\boldsymbol{x}} \rightarrow \boldsymbol{x} = \boldsymbol{T}^{-1}\tilde{\boldsymbol{x}}$$

$$\boldsymbol{T}\dot{\boldsymbol{x}} = \underline{\hspace{11cm}}$$

$$\boldsymbol{y} = \underline{\hspace{9.5cm}}$$

Description of Systems

State Space Representation

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$y = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

$\downarrow$ Laplace transform

$$\underline{\hspace{1cm}} = \underline{\hspace{3cm}}$$

$$\boldsymbol{X} = \underline{\hspace{5cm}}$$

$$\boldsymbol{Y} = \underline{\hspace{12cm}}$$

Newton-Euler Equations

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{D}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{B}\boldsymbol{u}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{q}$$

$\downarrow$ Laplace transform

$$\underline{\hspace{7cm}} = \underline{\hspace{1.2cm}}$$

$$\boldsymbol{Q}(s) = \underline{\hspace{7.5cm}}$$

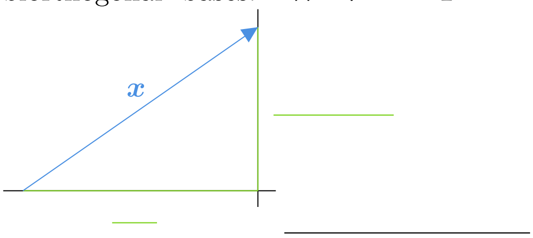
$$\boldsymbol{Y} = \underline{\hspace{7.5cm}}$$

Projector

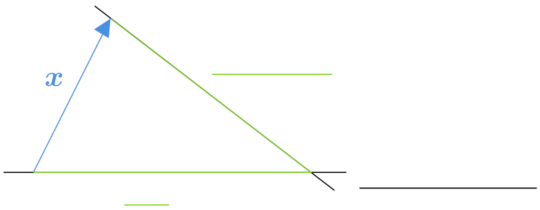
$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{x} + (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{x} \Rightarrow \mathbb{R}^n = \mathrm{span}(S_1\underline{\qquad\quad} + S_2\underline{\qquad\quad})$$

Galerkin

biorthogonal bases: $\boldsymbol{W}^T\boldsymbol{V} = \boldsymbol{I}$



Petrov-Galerkin



Reduction Approach

State Space Representation

$\boldsymbol{\Sigma} \overset{\boldsymbol{P}}{\rightsquigarrow} \bar{\boldsymbol{\Sigma}}$

$\frac{d}{dt}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$

$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$

$\boldsymbol{x} \approx \boldsymbol{V}\bar{\boldsymbol{x}}$

Newton-Euler Equations

$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{D}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{B}\boldsymbol{u}$

$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{q}$

$\boldsymbol{q} \approx \boldsymbol{V}\bar{\boldsymbol{q}}$

1.2 Linear Reduction Methods

1.2.1 Modal Reduction

1. ____

   $\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{D}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = 0$ (homogeneous) (Assumpt: Proportional Damping)

   $\boldsymbol{q}(t) = \boldsymbol{e}^{\lambda t}\boldsymbol{\varphi}$

   ↳ $(\lambda^2\boldsymbol{M} + \lambda\boldsymbol{D} + \boldsymbol{K})\boldsymbol{\varphi} = 0$ QEVP (solve with generalized Eigenproblem)

   $\rightarrow$ $\boldsymbol{N}$ eigenvalues $-\lambda_1^2, ..., -\lambda_n^2,$ $\lambda_i^2 < 0$

   eigenfrequencies $\omega_i > 0$ as $w_i = \sqrt{-\lambda_i^2} \Leftrightarrow \lambda_i = \pm i\omega_i$

   $\rightarrow$ $\boldsymbol{N}$ Eigenvectors real $\boldsymbol{\varphi}_i$ are eigenmodes

   $\rightarrow$ $\boldsymbol{\varphi}_i$ mass orthogonalized $\Leftrightarrow \Phi^T\boldsymbol{M}\Phi = I$

2. ______________________________

   $\underbrace{\Phi^T\boldsymbol{M}\Phi}_{I}\ddot{\bar{\boldsymbol{q}}} + \underbrace{\Phi^T\boldsymbol{D}\Phi}_{diag(2w_i\xi_i)}\ \bar{\boldsymbol{q}} + \underbrace{\Phi^T\boldsymbol{K}\Phi}_{diag(w_i^2)}\bar{\boldsymbol{q}} = \Phi^T\boldsymbol{B}\boldsymbol{u}$ only diagonal matrices
   $\rightarrow$ uncoupled system of ODES

   $y = \boldsymbol{C}\Phi\bar{\boldsymbol{q}}$

   ↳ $\bar{\boldsymbol{H}}(i\omega) = \sum_{i=1}^{N} \frac{[\bar{\boldsymbol{C}}]_{*i}\ [\bar{\boldsymbol{B}}]}{i*\omega_i^2 - \omega^2 + 2i\omega\omega_i\xi_i}$ "summand big for ω close to ω_i"

3. ______________

   $\boldsymbol{V} = \boldsymbol{W} = [\varphi_1, ..., \varphi_n],$

   Thumb rule
   φ_i to $\omega_i \approx 2\omega_{int}$
   $\rightarrow$ truncate uncontr./ unobs. eigenmodes

4. _________________

   $$\boldsymbol{HSV} : \sigma_i^2 \approx \frac{\|\ [C]_{xi}\ \|_2 \|\ [B]_{ix}\ \|_2}{4\omega_i\xi_i}$$

   approximate contribution of mode φ_i to the $\boldsymbol{H}_2$ and $\boldsymbol{H}_\infty$ norm.

Remarks:

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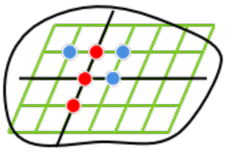
1.2.2 Craig-Bampton Reduction

1.



$$\hookrightarrow \underbrace{\begin{bmatrix} \boldsymbol{M}_{bb} & \boldsymbol{M}_{b_i} \\ \boldsymbol{M}_{ib} & \boldsymbol{M}_{ii} \end{bmatrix}}_{\boldsymbol{P}^T\boldsymbol{M}\boldsymbol{P}} \begin{bmatrix} \ddot{\boldsymbol{q}}_b \\ \ddot{\boldsymbol{q}}_i \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_{bb} & \boldsymbol{K}_{bi} \\ \boldsymbol{K}_{ib} & \boldsymbol{K}ii \end{bmatrix} \underbrace{\begin{bmatrix} \boldsymbol{q}_b \\ \boldsymbol{q}_i \end{bmatrix}}_{\boldsymbol{P}\boldsymbol{q}} = \begin{bmatrix} \boldsymbol{B}_b \\ \boldsymbol{B}_i \end{bmatrix} \boldsymbol{u}$$

$$\boldsymbol{y} = [\boldsymbol{C}_b \ \boldsymbol{C}_i] \begin{bmatrix} \boldsymbol{q}_b \\ \boldsymbol{q}_i \end{bmatrix}$$

2.

$$\boldsymbol{H}(0) = \boldsymbol{C}[\cancel{s^2\boldsymbol{M}} + \boldsymbol{K}]^{-1}\boldsymbol{B}$$

$$\boldsymbol{H}(0) = [\boldsymbol{C}_b\boldsymbol{C}_i] \begin{bmatrix} \boldsymbol{K}_{bb} & \boldsymbol{K}_{bi} \\ \boldsymbol{K}_{ib} & \boldsymbol{K}_{ii} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{B}_b \\ \boldsymbol{B}_i \end{bmatrix}$$

3.

$$\boldsymbol{H}(0) = \boldsymbol{C}_i \ \boldsymbol{K}_{ii}^{-1} \ \boldsymbol{B}_i + \hat{\boldsymbol{C}}_b \ \hat{\boldsymbol{K}}_{bb}^{-1} \ \hat{\boldsymbol{B}}_b$$

4.

$$\boldsymbol{V}_{corr} = \begin{bmatrix} I \\ -\boldsymbol{K}_{ii}^{-1}\boldsymbol{K}_{ib} \end{bmatrix} \in \mathbb{R}^{Nxb}$$

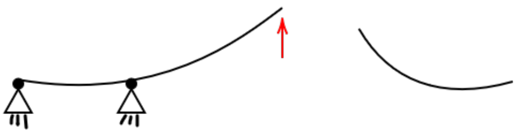
$$\xRightarrow{\text{static compensation}} \boldsymbol{H}(0) = \bar{\boldsymbol{H}}(0)$$

arbitrary frequency:

$$\boldsymbol{V}_{corr} = \begin{bmatrix} \boldsymbol{I} \\ -(-\omega^2 \boldsymbol{M}_{ii} + \boldsymbol{K}_{ii})^{-1}(-\omega^2 \boldsymbol{M}_{ib} + \boldsymbol{K}_{ib}) \end{bmatrix}$$

$$\boldsymbol{H}(\hat{s} = i\omega) = \bar{\boldsymbol{H}}(\hat{s} = i\omega)$$

Add correction modes to eigenmodes



1.2.3 Krylov Reduction

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1. ____________

$$\boldsymbol{T}_0^{(s_k)} = \boldsymbol{D} + \boldsymbol{C}(s_k\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$$
$$\boldsymbol{T}_j^{(s_k)} = \boldsymbol{C}((s_k\boldsymbol{I} - \boldsymbol{A})^{-1})^j(s_k\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$$

2. $\boldsymbol{H}(s) = \boldsymbol{T}_0^{(s_k)}\ s^0 + \boldsymbol{T}_1^{(s_k)}s^1 + \boldsymbol{T}_2^{(s_k)}s^2 + ... = \sum_{j=0}^{\infty}\boldsymbol{T}_j^{(s_k)}s^j$

3. __________

$\bar{\boldsymbol{H}}(s) = \sum_{j=0}^{r-1}\boldsymbol{T}_j^{(sk)}s^j \rightarrow$ numerical instable

4. ____

$$\mathcal{K}_j(\boldsymbol{M},\boldsymbol{R}) = span\ (\boldsymbol{R}, \boldsymbol{M}\boldsymbol{R}, \boldsymbol{M}^2\boldsymbol{R}, ..., \boldsymbol{M}^{j-1}\boldsymbol{R})$$

Input Krylov Space		Output Krylov Space
$\boldsymbol{R} = (\boldsymbol{A} - s_k\boldsymbol{I})^{-1}\boldsymbol{B}$	starting matrix	$\boldsymbol{R} = (\boldsymbol{A} - s_k\boldsymbol{I})^{-H}\boldsymbol{C}$
$\boldsymbol{M} = (\boldsymbol{A} - s_k\boldsymbol{I})^{-1}$	action matrix	$\boldsymbol{M} = (\boldsymbol{A} - s_k\boldsymbol{I})^{-H}$
span$(\boldsymbol{V}) =$		span$(\boldsymbol{W}) =$
$\mathcal{K}_{jb}^{s_k}((\boldsymbol{A} - s_k\boldsymbol{I})^{-1}, (\boldsymbol{A} - s_k\boldsymbol{I})^{-1}\boldsymbol{B})$	for multible s_k	$\mathcal{K}_{jc}^{s_k}((\boldsymbol{A} - s_k\boldsymbol{I})^{-H}, (\boldsymbol{A} - s_k\boldsymbol{I})^{-H}\boldsymbol{C})$

Then $\bar{\boldsymbol{T}}_j^{s_k} = \boldsymbol{T}_j^{s_k}$ for $\boldsymbol{j} = 0, ..., \boldsymbol{j}_b - 1 + \boldsymbol{j}_c - 1$

Solved with Arnoldi Algorithm: $(\boldsymbol{M},\boldsymbol{R}) \rightarrow \boldsymbol{V} = \boldsymbol{W}$ with orthogonal bases (Gram Schmidt)

5. ____________

   - $(s_k\boldsymbol{I} - \boldsymbol{A})$ must not be singular
   - for mechanical system $s_k = 0 + i\ {s_k}^{im}$
   - Better results for more s_k and more moments $\boldsymbol{j}$

2nd Order Krylov

$\mathcal{K}_j(\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{R}) = \text{colspan}\ \{\boldsymbol{P}_0, \boldsymbol{P}_1, ..., \boldsymbol{P}_{j-1}\}$

$\boldsymbol{P}_0 = \boldsymbol{R}$ $\qquad \boldsymbol{P}_1 = \boldsymbol{M}_1\boldsymbol{R}$

$\boldsymbol{P}_i = \boldsymbol{M}_1\boldsymbol{P}_{i-1} + \boldsymbol{M}_2\boldsymbol{P}_{i-2}$ $\qquad \boldsymbol{i} = 2, 3, ...$

$\Rightarrow$ second order Arnoldi $\Rightarrow \boldsymbol{V}$

Remarks:

+ ______________________ - ____________________________

+ ________________ - __________

+ ___________________________ - _____________________________________

+ ___________

1.2.4 Mirror images of poles

$\bar{\boldsymbol{H}}$ best approx of $\boldsymbol{H}$ with $\bar{\boldsymbol{\Sigma}}$ has simple poles $\bar{\lambda}_1, ..., \bar{\lambda}_n$.
Then $\bar{\boldsymbol{H}}(-\bar{\lambda}_k) = \boldsymbol{H}(-\lambda_k)$ and $\bar{\boldsymbol{H}}^{‘}(-\bar{\lambda}_k) = \boldsymbol{H}^{‘}(-\bar{\lambda}_k)$
Problem $\bar{\lambda}_K$ not known a prior! $\Rightarrow$ iterative algorithm

1.2.5 IRKA algo

1. ________________________________ _

2. ___________ __________

3. __________________ _________

reduced order:
$n =$ ____

1.2.6 Balanced Truncation

	Reachability/ Controllability	**Observability**
	$\bar{\boldsymbol{x}}$ reachable $\Leftrightarrow$ ____________________	$\bar{\boldsymbol{x}}$ not obs $\Leftrightarrow$ __________ __________
	$\bar{\boldsymbol{x}}$ contr. $\Leftrightarrow$ __________	
	$R(\boldsymbol{A}, \boldsymbol{B}) =$ ____________________	$\boldsymbol{O}(\boldsymbol{A}, \boldsymbol{C}) =$ ____________________
	$\boldsymbol{P}(t) := \int_0^t e^{A\tau} \boldsymbol{B}\boldsymbol{B}^H e^{A^H\tau} d\tau \underset{\lim\limits_{t\to\infty}}{\to} \boldsymbol{P}$	$\boldsymbol{Q}(t) := \int_0^t e^{A^H\tau} \boldsymbol{C}^H \boldsymbol{C} e^{A^H\tau} d\tau \underset{\lim\limits_{t\to\infty}}{\to} \boldsymbol{Q}$
	$X_{reach} = image(\boldsymbol{R}) = image(\boldsymbol{P}(t))$	$X_{unobs} = Kern(\boldsymbol{O}) = Kern(\boldsymbol{Q}(t))$
energy	$\vert\vert\bar{\boldsymbol{P}}\vert\vert^2 = \bar{\boldsymbol{x}}^H \boldsymbol{P}(\boldsymbol{t})^{-1} \bar{\boldsymbol{x}}$ for $\boldsymbol{t}_2 > \boldsymbol{t}_1$	$\bar{\boldsymbol{x}}^H \ \boldsymbol{Q}(t)\bar{\boldsymbol{x}}$
	$\boldsymbol{P}(\boldsymbol{t}_2) \geq \boldsymbol{P}(\boldsymbol{t}_1)$	
min. energy	$\bar{\boldsymbol{x}}^H \boldsymbol{P}^{-1} \bar{\boldsymbol{x}}$	$\bar{\boldsymbol{x}}^H \boldsymbol{Q} \bar{\boldsymbol{x}}$

Concept of BT Remove states $\bar{\boldsymbol{x}}$ with

- __________
- ________

$\to$ remove $\bar{\boldsymbol{x}} \ \widehat{=}$ eigenvector to small $\lambda(\boldsymbol{P})$ or $\lambda(\boldsymbol{Q})$ since $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$.

Balanced transformation: $\boldsymbol{P}, \boldsymbol{Q} \xrightarrow{T}$ ____________________

$\boldsymbol{HSV}$

$$\sigma_i = \sigma_i(\Sigma) := \sigma(\boldsymbol{H}) = \sqrt{\lambda_i(\boldsymbol{PQ})}$$

$$\sigma_i(\tilde{\Sigma}) = \sigma_i(\Sigma)$$

Singular values are preserved

$$\tilde{\boldsymbol{P}} = \boldsymbol{T}\boldsymbol{P}\boldsymbol{T}^H \qquad \tilde{\boldsymbol{Q}} = (\boldsymbol{T}^{-1})^H \boldsymbol{Q} \boldsymbol{T}^{-1}$$

Calc $\boldsymbol{T}$ and truncate to get $\boldsymbol{V}$ and $\boldsymbol{W}$

1. ______________________________
2. ________________________
3. ________________
4. ____
5. ________________________

$\Lsh$ $\bar{\Sigma} = \left[\begin{array}{c|c} \boldsymbol{W}^H\boldsymbol{A}\boldsymbol{V} & \boldsymbol{W}^H B \\ \hline \boldsymbol{C}\boldsymbol{V} & \boldsymbol{D} \end{array}\right]$

Remark:

+ ____
+ ____________________________________
+ ____________________________________
+ ____
- ______________

Calculate $\boldsymbol{Q}$ and $\boldsymbol{P}$ - numerical issue

Lyapunov eq: $\boldsymbol{A}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T = 0 \quad \boldsymbol{A}^T\boldsymbol{Q} + \boldsymbol{Q}\boldsymbol{A} + \boldsymbol{C}^T\boldsymbol{C} = 0$

$\Lsh$ ADI algorithm calculate $\boldsymbol{P}$, $\boldsymbol{Q}$ iteratively

$\Lsh$ low rank ADI approx $\boldsymbol{P}$, $\boldsymbol{Q}$ as low rank

BT for 2nd Order Mech Sys.

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_P & * \\ * & \boldsymbol{P}_V \end{bmatrix} \quad \boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_P & * \\ * & \boldsymbol{Q}_V \end{bmatrix} \in \mathbb{R}^{2N \times 2N}$$

$\sigma_P, \sigma_V, \sigma_{PV}, \sigma_{VP}$ only the Gramian can be balanced