Exercise Sheet 5

Exercise 7: Reachability and Observability

Given is the dynamical system Σ

$$\begin{aligned} \frac{d}{dt}\boldsymbol{x}(t) &= \boldsymbol{A} \cdot \boldsymbol{x}(t) + \boldsymbol{B} \cdot u(t), \quad t \in [0, T] \\ y(t) &= \boldsymbol{C} \cdot \boldsymbol{x}(t) \end{aligned}$$

with $\boldsymbol{x}(t) \in \mathbb{R}^N$, $u(t) \in \mathbb{R}$ and the system matrices

$$\boldsymbol{A} = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$
$$\boldsymbol{C} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \end{pmatrix}.$$

(a) Compute the reachable subspace X_{reach} of Σ.

(b) Compute the unobservable subspace X_{unobs} of Σ.

Hint: Use without proof that the unique solution of the dynamical system with initial state $\boldsymbol{x}_0$ and input $u(t)$ is given by

$$\boldsymbol{x}(t) = \exp(\boldsymbol{A}t) \cdot \boldsymbol{x}_0 + \int_0^t \exp(\boldsymbol{A}(t-\tau)) \cdot \boldsymbol{B}u(\tau)d\tau.$$

Exercise 8: Balanced Truncation

Let $\bar{\Sigma}_k$ be a system after applying balanced truncation to Σ. Show that $\bar{\Sigma}_k$ is indeed balanced, i.e., $\bar{\boldsymbol{P}} = \bar{\boldsymbol{Q}}$.

Hints

- Use the Lyapunov equations (see lecture). They have always a unique solution.
- Use that the transformed (by not yet reduced) system $\tilde{\Sigma}$ is balanced with $\tilde{\boldsymbol{P}} = \tilde{\boldsymbol{Q}} = \boldsymbol{\Lambda}$.

Notes

- The homework is due on Sunday night February 2th, 2025 at 11:59 p.m.