

## EXERCISE SHEET 5

**Exercise 7: Reachability and Observability**

Given is the dynamical system  $\Sigma$

$$\begin{aligned}\frac{d}{dt}\mathbf{x}(t) &= \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot u(t), \quad t \in [0, T] \\ y(t) &= \mathbf{C} \cdot \mathbf{x}(t)\end{aligned}$$

with  $\mathbf{x}(t) \in \mathbb{R}^N$ ,  $u(t) \in \mathbb{R}$  and the system matrices

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$
$$\mathbf{C} = \left( \frac{1}{2} \quad \frac{1}{2} \quad 0 \quad \cdots \quad 0 \right).$$

- (a) Compute the reachable subspace  $X_{\text{reach}}$  of  $\Sigma$ .
- (b) Compute the unobservable subspace  $X_{\text{unobs}}$  of  $\Sigma$ .

Hint: Use without proof that the unique solution of the dynamical system with initial state  $\mathbf{x}_0$  and input  $u(t)$  is given by

$$\mathbf{x}(t) = \exp(\mathbf{A}t) \cdot \mathbf{x}_0 + \int_0^t \exp(\mathbf{A}(t - \tau)) \cdot \mathbf{B}u(\tau)d\tau.$$

**Exercise 8: Balanced Truncation**

Let  $\bar{\Sigma}_k$  be a system after applying balanced truncation to  $\Sigma$ . Show that  $\bar{\Sigma}_k$  is indeed balanced, i.e.,  $\bar{\mathbf{P}} = \bar{\mathbf{Q}}$ .

Hints

- Use the Lyapunov equations (see lecture). They have always a unique solution.
- Use that the transformed (by not yet reduced) system  $\tilde{\Sigma}$  is balanced with  $\tilde{\mathbf{P}} = \tilde{\mathbf{Q}} = \mathbf{\Lambda}$ .

Notes

- The homework is due on Sunday night February 5th, 2023 at 11:59 p.m.