ON THE USE OF LINEAR COMPLEMENTARITY PROBLEMS FOR CONTACT OF PLANAR FLEXIBLE BODIES

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Abstract. In contrast to the finite element method (FEM) which formulates contact problems considering all local effects involved, there are also approaches which do not intend to consider local details. Linear complementarity problems result from formulations of unilateral contact problems as mathematical problems which yield exact solutions to contact and impact including friction. They are frequently used in the case of unilateral contact of rigid bodies in multibody systems and are discussed in detail in the literature by many researchers but there still are open questions in formulating these approaches for the case of deformable bodies.

This paper presents a formulation which extends these methods in order to consider the unilateral contact of constrained and non-constrained planar deformable bodies. In doing so, kinematic relationships governing the behavior of contact are formulated in such a way that they consider the effect of deformations. As a general approach for flexible multibody systems, the moving frame of reference approach and modal coordinates are used to describe deformable bodies. In this formulation the effects of deformation are taken into account starting from the relative velocity of contact points in the normal direction of contact and then the procedure is followed by introducing the relative acceleration of contact points which includes all the necessary terms needed for considering deformations. Then the complementarity relations are reformulated following the same procedure as for rigid bodies contact. Therefore, the main change of this algorithm compared to the rigid body case is in the formulation of the kinematics of contact.
1 INTRODUCTION

In the simulation of multibody systems in industry and engineering, there are many applications which deal with contact problems and make their modeling an essential and very demanding topic in multibody dynamics. Therefore, a lot of research focuses on this topic and many theoretical and mathematical methods are developed for this purpose.

With increasing availability of fast computers, special attention was paid to the numerical investigation of contact problems and their application. Meanwhile, more and more approaches have been developed and taken into service. These approaches are based on the physics of contact and take into account some assumptions in order to model the contact problem physically correct. Among these approaches, it may be pointed out that the FEM is the most powerful general numerical method in contact modeling of elastic bodies but its very high computational effort makes it not applicable to problems like the ones often described by multibody systems where one needs to simulate the contact problem over a long period of time.

Approaches based on penalty terms have been used frequently for contacts in multibody dynamics but they have some restrictions and are not always applicable.

Some other contact modeling approaches belong to another category and yield a mathematical formulation based on the kinematics of contacting bodies in a linear complementarity form. By solving these linear complementarity problems (LCPs) the exact solution to the contact problem can be obtained, see [6, 7].

Many researchers use this type of formulation for contact modeling of rigid bodies due to its capabilities in handling of frictional contact, including many sliding and sticking contact transitions. This has made it a powerful and useful approach in this field. In this approach, two subproblems must be formulated, the continual contact and impact.

Mathematicians are interested to investigate the LCPs from the mathematical point of view. In some of their formulations, the basic theory of convolution complementarity problems is given in which they deal with impact problems for elastic bodies with Coulomb friction [13].

Some others are more interested to investigate the implementation of contact in LCP form and its applications in engineering. As a result, some algorithms were developed based on unilateral constraints of the kinematics of contact points [1, 6, 8]. Initially, only planar relative contact kinematics was considered and then this algorithm was extended for the case of spatial contact, see [4].

In applications where the flexibility of contacting bodies is not negligible, rigid body contact modeling can not be used and the deformability of contacting bodies must be taken into account. Therefore, in this work, it has been tried to present an approach by reformulating the kinematic equations governing rigid body contact in order to be able to consider the deformation of contacting bodies. In doing so, the moving frame of reference approach is utilized in order to introduce the elastic coordinates into the equations of motion, see [11], [12]. It is important to emphasize that this paper just considers the continual contact case of deformable bodies and for impact calculation this formulation has to be extended.

An extended formulation for the continual contact of elastic bodies will be presented in Section 2. This procedure will start by reformulating the kinematics of contact for flexible bodies and will follow by introducing this in the complementarity form known from rigid body formulations. In Section 3 a comparison between the presented formulation and the rigid body formulation is given. Then, some results of simulated examples will be presented in order to show the feasibility of the described approach in Section 4. The paper will end with a conclusion and a list of references.
2 CONTACT FORMULATION

Deformable bodies are modeled here using the well-known moving frame of reference approach. By introducing rigid and elastic coordinates, the movement of bodies is separated into two independent parts, the rigid body movement and the small elastic deformation. In this approach, the deformation of bodies is assumed to be small compared to the dimension of bodies, but not negligible.

2.1 KINEMATICS OF CONTACT POINTS

In Figure 1, two deformable bodies \(i\) and \(j\) are depicted, which are in contact in point \(k\). In order to describe the position and orientation of these deformable bodies with respect to the global coordinate system, two sets of generalized coordinates \(q_i = (R_i, \theta_i, q_{f,i})\) and \(q_j = (R_j, \theta_j, q_{f,j})\) including the rigid \((R, \theta)\) and elastic \((q_f)\) generalized coordinates are used. The rigid ones specify the position \((R)\) and orientation \((\theta)\) of the body reference coordinate systems with respect to the global coordinate system. In addition, the elastic coordinates are defined with respect to the body reference coordinate systems and are supposed to specify the position and orientation of any point on the bodies locally. Later all generalized coordinates are summarized in \(q\). However, \(q\) is not a vector of minimal coordinates since constraints from joints, ... must be considered by Lagrangian multipliers additionally.

![Figure 1: Contact between two deformable bodies](image)

Considering one deformable body \(i\), one can calculate the velocity of each arbitrary point \(p_i\) located on the body \(i\) from

\[
v^{p_i} = \dot{R}_i + \dot{A}_i \cdot \bar{u}_i + A_i \cdot \dot{\bar{u}}_i,
\]

where \(A_i\) denotes the transformation matrix of body \(i\) and \(\bar{u}_i\) is the position vector of the point \(p_i\) in the reference coordinate system of body \(i\) in the deformed configuration, see [11]. By substituting the velocities \(\dot{\bar{u}}_i = S_i \cdot \dot{q}_f\), the above equation can be written as

\[
v^{p_i} = \dot{R}_i + \dot{A}_i \cdot \bar{u}_i + A_i \cdot S_i \cdot \dot{q}_f.
\]

In this relation, \(S_i\) is the shape matrix of body \(i\) which is independent of time. One can write the second term of the right-hand side of this equation in terms of the derivative of the orientation...
\( \theta_i \) with respect to time as
\[
\dot{A}_i \cdot \ddot{u}_i = B_i \cdot \dot{\theta},
\]
where \( B_i \) is a \( 2 \times 1 \) matrix in the case of planar flexible bodies. It is defined in terms of the partial derivative of the transformation matrix \( A_i \) times the local position vector \( \ddot{u}_i \) of the point \( p_i \) with respect to the rotational coordinates \( \theta_i \), see [11].

\[
B_i = \frac{\partial}{\partial \theta_i} \left( A_i \cdot (\ddot{u}_{0i} + S_i \cdot \dot{q}_i) \right).
\]

In this relation, the vector \( \ddot{u}_{0i} \) denotes the position of point \( p_i \) on the undeformed body \( i \). Using Eq. (3), Eq. (2) can be written in terms of the generalized velocity vector \( \dot{q}_i \) as
\[
\mathbf{v}^{pi} = L_i^{pi} \cdot \dot{q}_i.
\]

In this relation, the vector \( \dot{q}_i \) includes rigid and elastic generalized velocities \( \dot{q}_i = (\dot{R}_i, \dot{\theta}_i, \dot{q}_f) \) and \( L_i \) is a matrix which projects the generalized velocity vector \( \dot{q}_i \) onto the velocities in the global coordinate system
\[
L_i = \begin{bmatrix} I & B_i & A_i \cdot S_i \end{bmatrix},
\]
where \( I \) is the \( 2 \times 2 \) identity matrix.

According to Figure 1 and noticing that at any contact point \( k \) the normal vectors \( (n_i, n_j) \) and tangential vectors \( (t_i, t_j) \) of body \( i \) and body \( j \) are aligned with each other, the relative velocities in normal and tangential directions for contact point \( k \) can be written as
\[
\dot{g}_N^k = n_i^k \cdot (L_i^k \cdot \dot{q}_i - L_j^k \cdot \dot{q}_j), \quad \dot{g}_T^k = t_i^k \cdot (L_i^k \cdot \dot{q}_i - L_j^k \cdot \dot{q}_j).
\]

According to these relationships, for any contact point \( k \) between bodies \( i \) and \( j \), the matrices \( L_i^k \) and \( L_j^k \) must be calculated since they depend on the positions.

The relative accelerations in the normal and tangential directions for contact point \( k \) are calculated by taking the derivative of the relative velocities
\[
\ddot{g}_N^k = (W_N^k)^T \cdot \ddot{\dot{q}}_{ij} + (w_N^k)^T \cdot \dot{q}_{ij}, \quad \ddot{g}_T^k = (W_T^k)^T \cdot \ddot{\dot{q}}_{ij} + (w_T^k)^T \cdot \dot{q}_{ij},
\]
where
\[
\ddot{q}_{ij} = \begin{bmatrix} \ddot{q}_i \\ \ddot{q}_j \end{bmatrix}, \quad (W_N^k)^T = \begin{bmatrix} (n_i^k \cdot L_i^k)^T \\ -(n_i^k \cdot L_j^k)^T \end{bmatrix}, \quad (W_T^k)^T = \begin{bmatrix} (t_i^k \cdot L_i^k)^T \\ -(t_i^k \cdot L_j^k)^T \end{bmatrix},
\]
\[
\begin{align*}
(w_N^k)^T &= \begin{bmatrix} (n_i^k \cdot L_i^k + n_j^k \cdot L_j^k)^T \\ -(n_i^k \cdot L_j^k + n_j^k \cdot L_i^k)^T \end{bmatrix}, \\
(w_T^k)^T &= \begin{bmatrix} (t_i^k \cdot L_i^k + t_j^k \cdot L_j^k)^T \\ -(t_i^k \cdot L_j^k + t_j^k \cdot L_i^k)^T \end{bmatrix}.
\end{align*}
\]

It is clear that effects of deformabilities are introduced to the kinematic relation of the relative normal and tangential accelerations \( (\ddot{g}_N, \ddot{g}_T) \) of contact point \( k \) through the matrices \( L_i^k \) and \( L_j^k \). Although Eq. (3) looks very similar to the rigid body case, its computation is very different.

This equation which holds for contact point \( k \) can be used to obtain the matrix form of the relative normal and tangential accelerations \( (\ddot{g}_N, \ddot{g}_T) \) for all \( n_c \) contact points between \( n_b \) bodies
\[
\ddot{g}_N = W_N^T \cdot \ddot{q} + w_N^T \cdot \dot{q}, \quad \ddot{g}_T = W_T^T \cdot \ddot{q} + w_T^T \cdot \dot{q},
\]
where

\[ W_N = \begin{bmatrix} W^1_N & \cdots & W^k_N & \cdots & W^{n_e}_N \end{bmatrix}, \quad W_T = \begin{bmatrix} W^1_T & \cdots & W^k_T & \cdots & W^{n_e}_T \end{bmatrix}, \]

\[ w_N = \begin{bmatrix} w^1_N & \cdots & w^k_N & \cdots & w^{n_e}_N \end{bmatrix}, \quad w_T = \begin{bmatrix} w^1_T & \cdots & w^k_T & \cdots & w^{n_e}_T \end{bmatrix}, \]

and the vectors \( \dot{q} \) and \( \ddot{q} \) are the generalized velocities and accelerations of the system consisting of all \( n_b \) bodies.

Then, the relative accelerations \( \ddot{g}_N \) and \( \ddot{g}_T \) and the equations of motion of flexible bodies will be used together in order to form the complementarity relationships between the contact forces and the relative accelerations. This procedure will be described next.

### 2.2 FLEXIBLE BODIES IN MULTIBODY SYSTEMS

In this section, the equations of motion of one flexible body \( i \) taking into account the contact forces in a system of interconnected rigid and flexible bodies are written and then the equations of motion of flexible multibody systems are given in matrix form.

Considering a flexible body \( i \), one can write its nonlinear equations of motion as

\[
M_i \cdot \ddot{q}_i + C_i \cdot \dot{q}_i + K_i \cdot q_i + \left( \frac{\partial c}{\partial q_i} \right)^T \cdot \lambda_i = F_{ext_i} + F_{v_i} + F_{c_i}. \tag{10}
\]

Here \( M_i \) is the mass matrix of body \( i \), \( C_i \) and \( K_i \) are the damping and stiffness matrices arising from the elastic coordinates, \( \frac{\partial c}{\partial q_i} \) is the Jacobian matrix containing the derivatives of all constraints \( c = 0 \) except the contact constraints, \( \lambda_i \) are the Lagrangian multipliers corresponding to the constraint forces, \( F_{ext_i} \) is the vector of generalized external forces, \( F_{v_i} \) is the vector of generalized Coriolis forces and \( F_{c_i} \) is the vector of generalized contact forces. These equations can be summarized for the whole system together with the second derivatives of the constraints in matrix form for flexible multibody systems including constrained and non-constrained bodies, see [11],

\[
\begin{bmatrix}
M & C_q \\
C_q & 0
\end{bmatrix} \begin{bmatrix}
\ddot{q} \\
\dot{\lambda}
\end{bmatrix} = \begin{bmatrix}
F_{ext} + F_v - C \cdot \dot{q} - K \cdot q \\
-\frac{\partial c}{\partial q} \cdot \dot{q} \cdot \dot{q} - 2 \frac{\partial^2 c}{\partial q \partial t} \cdot \dot{q} - \frac{\partial^2 c}{\partial t \partial t} \end{bmatrix} + \begin{bmatrix}
F_c \\
0
\end{bmatrix}. \tag{11}
\]

When two bodies come in contact, normal and tangential contact forces arise as the result of the collision. Therefore, the contact force vector \( F_c \) in Eq. (11) for \( n_c \) contact points can be supposed to be the summation of normal and tangential forces which is written in terms of two different vectors \( \lambda_N \) and \( \lambda_T \), for details please refer to [10],

\[
F_c = \left( \begin{bmatrix} W^1_N \cdots W^{n_e}_N \end{bmatrix} \right) \cdot \begin{bmatrix} \lambda^1_N \\
\vdots \\
\lambda^{n_e}_N \end{bmatrix} + \left( \begin{bmatrix} W^1_T \cdots W^{n_e}_T \end{bmatrix} \right) \cdot \begin{bmatrix} \lambda^1_T \\
\vdots \\
\lambda^{n_e}_T \end{bmatrix} = W_N \cdot \lambda_N + W_T \cdot \lambda_T. \tag{12}
\]

By separating the tangential contact forces into the two different cases of sliding and sticking contact, one can rewrite Eq. (12) to get

\[
F_c = \left( \begin{bmatrix} W_{N1} + W_{G1} \cdot \mu_{G1} \cdots W_{Nn_c} + W_{Gn_c} \cdot \mu_{Gn_c} \end{bmatrix} \right) \cdot \lambda_N + \left( \begin{bmatrix} W_{H1} \cdots W_{Hn_c} \end{bmatrix} \right) \cdot \lambda_H. \tag{13}
\]
In this equation, which is similar to [6], \( W_G \) and \( W_H \) are matrices extracted from the matrix \( W_T \) which correspond to the sliding and sticking parts, respectively.

The equations of motion (11) can be reformulated by substituting the contact forces from Eq. (13)

\[
M_c \cdot \ddot{q}_c - h_c - \begin{bmatrix} W_N + W_G \cdot \mu_G & W_H \end{bmatrix} \begin{bmatrix} \lambda_N \\ \lambda_H \end{bmatrix} = 0.
\]

(14)

Now the equations of motion for multibody systems including constrained and non-constrained rigid and flexible bodies in terms of contact forces (which are denoted by \( \lambda_N \) and \( \lambda_H \)) have been obtained and this form of equations will be used in the next section to construct the complementarity form of the equations of motion with contact.

### 2.3 CONSTRUCTION OF THE COMPLEMENTARITY FORM

In this section, it has been tried to summarize a similar procedure as described in [6] and construct the complementarity equations for continual contact of deformable bodies by utilizing the equations obtained in the previous sections. Maybe at the first view, these equations seem to be very similar to the rigid ones, but as it is pointed out before, all the effect of deformabilities are taken into account inside the kinematic quantities and the calculation procedure of these quantities is totally different.

Starting from Eq. (14), supposing that there is no dependent constraint in the system and so \( M_c \) is a regular matrix, we can find the acceleration \( \ddot{q}_c \) as a function of the Lagrangian multipliers

\[
\ddot{q}_c = M_c^{-1} \cdot h_c + M_c^{-1} \cdot W_{NH} \cdot \lambda.
\]

(15)

Then, Eq. (9) is rewritten for the cases of sliding and sticking contact

\[
\ddot{g}_{NH} = \begin{bmatrix} \ddot{g}_N \\ \ddot{g}_H \end{bmatrix} = \begin{bmatrix} W_N & W_H \end{bmatrix}^T \cdot \dot{q} + \begin{bmatrix} w_N \\ w_H \end{bmatrix}^T \cdot \dot{q}.
\]

(16)

The vector \( w_H \) is the part of the vector \( w_T \) which corresponds to the sticking contact. By inserting the Lagrangian multipliers \( \lambda_c \), which denotes the constraint forces (except contact forces) into the unknown variables as we did for the generation of equations of motion in Eq. (11), the above relation is rewritten as

\[
\ddot{g}_{NH} = \begin{bmatrix} \ddot{g}_N \\ \ddot{g}_H \end{bmatrix} = \begin{bmatrix} W_N^T & 0 \\ W_H^T & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_c \\ \lambda_c \end{bmatrix} + \begin{bmatrix} w_N^T \cdot \dot{q} \\ w_H^T \cdot \dot{q} \end{bmatrix}.
\]

(17)

In the next step, the vector \( \ddot{q}_c \) from Eq. (15) can be substituted in this equation

\[
\ddot{g}_{NH} = W^T \cdot M_c^{-1} \cdot h_c + W^T \cdot M_c^{-1} \cdot W_{NH} \cdot \lambda + w.
\]

(18)

By following the same procedure as described in [6] and by using Eq. (18) one can construct the complementarity form of the equations of motion and then simulate the continual contact problem of deformable bodies, too. In doing so and in order to handle the condition of switching between sliding and sticking contact, the tangential part of contact forces must be decomposed into two different parts and, thereby, the frictional case of contact will be handled appropriately.
The final form of the complementarity equations based on our notation and according to the above mentioned points is

$$
\begin{bmatrix}
\ddot{\mathbf{g}} \\
\lambda_{H_0}
\end{bmatrix} = \begin{bmatrix} \mathbf{W}^T \cdot \mathbf{M}_c^{-1} \cdot \mathbf{W} \cdot \mathbf{N}_H & \mathbf{I}_0 \\
\mathbf{N}_H - \mathbf{I} & 0
\end{bmatrix} \cdot \begin{bmatrix} \mathbf{\lambda} \\
\mathbf{z}
\end{bmatrix} + \begin{bmatrix} \mathbf{W}^T \cdot \mathbf{M}_c^{-1} \cdot \mathbf{h}_c + \mathbf{w} \\
0
\end{bmatrix},
$$

$$
\begin{bmatrix}
\ddot{\mathbf{g}} \\
\lambda_{H_0}
\end{bmatrix} \geq 0, \\
\begin{bmatrix} \mathbf{\lambda} \\
\mathbf{z}
\end{bmatrix} \geq 0, \\
\begin{bmatrix} \ddot{\mathbf{g}} \\
\lambda_{H_0}
\end{bmatrix} \cdot \begin{bmatrix} \mathbf{\lambda} \\
\mathbf{z}
\end{bmatrix} = 0.
$$

(19)

The parameters $\lambda_{H_0}$, $\mathbf{N}_H$ and $\mathbf{z}$ in this equation, are chosen in the same way as in [6] and have the same meaning. These parameters are used in order to formulate the complementarity form of the equations in such a way to handle switching between sliding and sticking cases of contact.

3 COMPARISON BETWEEN THE ELASTIC AND THE RIGID FORMULATION

The presented formulation in this paper is applicable for continual contact modeling of all multibody systems including constrained and non-constrained rigid and flexible bodies supposing that the deformations are small. In the case of rigid bodies, this formulation leads to the same results as the formulation developed for rigid bodies in [6].

For a comparison between the obtained formulation for $\ddot{\mathbf{g}}$ for deformable bodies and the case of rigid bodies, one can start from the presented relationships and reach to the formulations in [6] by choosing the elastic coordinates as zero coordinates which denotes that there is no deformability in the system.

Considering two rigid bodies $i$ and $j$ and referring to Eq. (8), the relationship of $\ddot{\mathbf{g}}_N^k$ is

$$
\ddot{\mathbf{g}}_N^k = \mathbf{W}_N^k \cdot \mathbf{q}_{ij} + \mathbf{w}_N^k \cdot \mathbf{q}_{ij}.
$$

For simplicity and without loss of generality, we suppose that one of the bodies, for example body $j$, is the ground and, therefore, its corresponding coordinates vanish from the above equation. With this assumption, the above equation can be expanded in terms of its parameters

$$
\ddot{\mathbf{g}}_N^k = \mathbf{n}_i^k \cdot \mathbf{L}_i^k \cdot \mathbf{q}_i + (\mathbf{n}_i^k \cdot \mathbf{\dot{L}}_i^k + \mathbf{n}_i^k \cdot \mathbf{\ddot{L}}_i^k) \cdot \mathbf{q}_i.
$$

(20)

Since the body $i$ is assumed to be a rigid body, all its elastic coordinates vanish and from Eq. (4) it remains only $\mathbf{B}_i = \partial(\mathbf{A}_i \cdot \tilde{\mathbf{u}}_{ih})/\partial \mathbf{\theta}$. In the case of rigid planar systems, the matrix $\mathbf{L}_i^k$ is a $2 \times 3$ matrix and, therefore, the above equation is simplified to

$$
\ddot{\mathbf{g}}_N^k = \mathbf{n}_i^k \cdot (\mathbf{\dot{R}}_i + \dot{\mathbf{\theta}}_i \mathbf{B}_i^k + \mathbf{\ddot{\theta}}_i \mathbf{B}_i^k) + \mathbf{n}_i^k \cdot (\mathbf{\dot{R}}_i + \dot{\mathbf{\theta}}_i \mathbf{B}_i^k + \ddot{\mathbf{\theta}}_i \mathbf{B}_i^k) = \mathbf{n}_i^k \cdot \mathbf{a}_i^k + \mathbf{n}_i^k \cdot \mathbf{\dot{v}}_i^k.
$$

(21)

From this equation it is clear that the relative normal acceleration $\ddot{\mathbf{g}}_N$ for any contact point $k$ consists of two parts. The first part is due to the acceleration of that point in the direction of the normal vector $\mathbf{n}_i^k$ and the second part is due to the change in the direction of the normal vector.

This procedure can be followed exactly in the same way for the relative tangential accelerations $\ddot{\mathbf{g}}_T^k$ of the contact point $k$ and then extended for all contact points

$$
\ddot{\mathbf{g}}_T^k = \mathbf{t}_i^k \cdot \mathbf{L}_i^k \cdot \mathbf{q}_i + (\mathbf{t}_i^k \cdot \mathbf{\dot{L}}_i^k + \mathbf{\ddot{t}}_i^k \cdot \mathbf{L}_i^k) \cdot \mathbf{q}_i = \mathbf{t}_i^k \cdot (\mathbf{\dot{R}}_i + \dot{\mathbf{\theta}}_i \mathbf{B}_i^k + \mathbf{\ddot{\theta}}_i \mathbf{B}_i^k) + \mathbf{\ddot{t}}_i^k \cdot (\mathbf{\dot{R}}_i + \dot{\mathbf{\theta}}_i \mathbf{B}_i^k + \ddot{\mathbf{\theta}}_i \mathbf{B}_i^k) = \mathbf{t}_i^k \cdot \mathbf{a}_i^k + \mathbf{\ddot{t}}_i^k \cdot \mathbf{\dot{v}}_i^k.
$$

(22)

Therefore, one can easily verify that the presented relationships for relative accelerations are valid and the matrices $\mathbf{W}_N$, $\mathbf{W}_T$, $\mathbf{w}_N$ and $\mathbf{w}_T$ which are used to formulate the continual contact
modeling of deformable bodies in a complementarity form can take into account the effect of deformabilities. Compared to the continual contact of rigid bodies, these four matrices are the most important parameters which have to be reformulated in such a way to lead to a correct formulation for the continual contact of deformable bodies. The results shown in the following section confirm the validity of this procedure as well.

4 NUMERICAL EXAMPLES AND OBTAINED RESULTS

Next, the formulation presented in the previous sections is implemented and examined in two examples. With due attention to the numerical results of these examples, one can verify the validity and feasibility of the described approach. In these examples, the continual contact of an elastic rectangular block on a rigid foundation is investigated. In both examples, it is supposed that the elastic rectangular block slides on the foundation and after a while due to the effect of the friction force sliding contact changes to sticking contact (in the first example, this situation happens for the second case with choosing a larger value for the friction coefficient). The examples differs in two important points: first, their foundations have different shapes and second, in the second example the rectangular block is restricted by a constraint which forces the center of the block to follow a certain curve. These cases are explained briefly in the corresponding sections. The shape of the rigid foundations are chosen in such a way to activate all the important terms in the presented formulation.

4.1 Example 1: Elastic rectangular block on an inclined and curved rigid foundation

In Figure 2, an elastic rectangular block in an arbitrary position and orientation with respect to the global coordinate system $O$ is shown. The coordinate system $O_i$ is attached rigidly to node 4 of the elastic block and is considered to be the block reference coordinate system. The x-axis is aligned to edge (4,1). Thus, the rigid coordinates of the block $R$ and $\theta$ are calculated from position and orientation of this coordinate system. Therefore, the position of any arbitrary point on the elastic block can be calculated with respect to the coordinate system $O_i$ through the coordinates of the nodes. The selection of elastic coordinates is $q_f = (q_{f1_x}, q_{f2_x}, q_{f2_y}, q_{f3_x}, q_{f3_y})$ since they have to be compatible with the rigid coordinates. Rigid and elastic coordinates have to lead to a unique description of position and orientation of the elastic block. This means that node 1 can only move in the $x$ direction with respect to the coordinate system $O_i$. The vector of generalized coordinates for this example can be considered as $\mathbf{q} = (R_x, R_y, \theta, q_{f1_x}, q_{f2_x}, q_{f2_y}, q_{f3_x}, q_{f3_y})$.

The shape matrix of this elastic block is given by

$$S = \begin{bmatrix} N_1 & N_2 & 0 & N_3 & 0 \\ 0 & 0 & N_2 & 0 & N_3 \end{bmatrix},$$

(23)

where $N_1, N_2$ and $N_3$ are shape functions defined in terms of the length $L$ and the height $H$ of the block as follows

$$N_1 = \frac{x}{L} (1 - \frac{y}{H}), \quad N_2 = \frac{x}{L} \frac{y}{H}, \quad N_3 = (1 - \frac{x}{L}) \frac{y}{H}. \quad (24)$$

These shape functions have the usual property that they take the value one at their corresponding corner and the value zero at the other corners

$$\left\{ \begin{array}{l} \text{at node 1: } N_1 = 1, \quad N_2 = 0, \quad N_3 = 0, \\ \text{at node 2: } N_1 = 0, \quad N_2 = 1, \quad N_3 = 0, \\ \text{at node 3: } N_1 = 0, \quad N_2 = 0, \quad N_3 = 1, \\ \text{at node 4: } N_1 = 0, \quad N_2 = 0, \quad N_3 = 0. \end{array} \right. \quad (25)$$
Based on the selected coordinates and the described shape matrix, one can construct the matrix $L^k$ and start to generate the kinematical relationships of contact for an arbitrary contact point $k$. After some mathematical manipulations, the matrix $L^k$ for contact point $k$ can be given as

$$L^k = \begin{bmatrix} 1 & 0 & -x \sin \theta - y \cos \theta & N_1 \cos \theta & N_2 \cos \theta & -N_2 \sin \theta & N_3 \cos \theta & -N_3 \sin \theta \\ 0 & 1 & x \cos \theta - y \sin \theta & N_1 \sin \theta & N_2 \sin \theta & N_2 \cos \theta & N_3 \sin \theta & N_3 \cos \theta \end{bmatrix}. \quad (26)$$

In the next step, the derivative of this matrix with respect to time has to be taken and then the matrix $\dot{L}^k$ can be generated. Afterwards, following Eq. (8) all necessary parameters for contact between the elastic block and the rigid foundation are obtained.

After formulating the kinematical relationships of Eq. (8), the equations of motion of the elastic block on the rigid foundation have to be calculated. Since there is no explicit constraint in the system, the second row of the system of equations of Eq. (11) vanishes and this equation can be rewritten as

$$M \cdot \ddot{q} + C \cdot \dot{q} + K \cdot q = F_{ext} + F_v + F_c. \quad (27)$$

The matrices $M$, $C$ and $K$ in this relationship can be written in their expanded form as

$$M = \begin{pmatrix} m_{RR} & m_{R\theta} & m_{Rf} \\ m_{\theta R} & m_{\theta\theta} & m_{\theta f} \\ \text{symmetric} & m_{ff} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{ff} & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_{ff} \end{pmatrix}. \quad (28)$$

The damping matrix $c_{ff}$ corresponding to the elastic coordinates can often be chosen as Rayleigh damping in terms of the mass matrix $m_{ff}$ and the stiffness matrix $k_{ff}$ using two constant parameters $\alpha$ and $\beta$, see [9]

$$c_{ff} = \alpha m_{ff} + \beta k_{ff}. \quad (29)$$
The quantities in Eq. (27) are derived following the procedure of [11]. The contact force vector \( \mathbf{F}_c \) can be calculated based on the relations in Section (2.2) in terms of the Lagrangian multipliers \( \lambda_N \) and \( \lambda_H \).

In the last step, the complementarity form of the equations of motion is derived and by solving this complementarity equations, the contact of the elastic block on the rigid foundation can be simulated.

In this work, the linear complementarity equations are solved by the PATH solver which is an algorithm for mixed complementarity problems, see [2] and [3]. In order to overcome common problems of instability in the integration process of flexible bodies, the equations of motion are integrated using the RADAU5 code [5].

For the simulation of this academic test example, the following values are used:

initial conditions: \( x_0 = -13.66 \text{ m}, \ y_0 = -3.66 \text{ m}, \ \theta_0 = -30 \text{ Deg}, \ q_f = 0, \dot{x}_0 = 0, \dot{y}_0 = 0, \dot{\theta}_0 = 0, \dot{q}_f = 0 \),

material: \( E = 1000 \text{ N/m}, \ \nu = 0.3, \ \rho = 2 \text{ kg/m}^2 \),

geometry: \( H = 1 \text{ m}, \ L = 2 \text{ m} \).

The elastic block and the rigid foundation are depicted in Figure 3. According to this figure, the foundation consists of three parts: two inclined straight parts and a curved part which is an arc of a circle with radius \( r = 10 \text{ m} \). The length of each inclined part is \( 10 \text{ m} \) and the inclination angle is \( \alpha = 30 \text{ Deg} \). The simulation is done for \( t_{end} = 10 \text{ s} \) and for two cases with different friction coefficients \( \mu = 0.1 \) and \( \mu = 0.2 \). Some results of these simulations are illustrated in Figures 4 and 5.

![Figure 3: Sliding and sticking contact of the non-constrained elastic block on the rigid foundation](image-url)

In the first simulation (\( \mu = 0.1 \)), as it can be seen from Figure 4, the elastic block slides on the foundation, leaves the left inclined part of the path at \( t = 2 \text{ s} \) and enters the curved part. It reaches the lowest point of the curved path at \( t = 2.7 \text{ s} \) where \( x = 0 \text{ m}, \ y = -10 \text{ m} \) and \( \theta = 0 \text{ rad} \) and then enters the right inclined part at \( t = 3.8 \text{ s} \). There, \( \theta \) has a constant value \( 0.523 \text{ rad} \) which is equal to the inclination angle of the right inclined part. The elastic block moves upward on the right inclined path and stops for a moment at \( t = 4.2 \text{ s} \) before it starts to move in opposite direction. This point can also be seen from Figure 4a where the \( x \) and \( y \) coordinates of the reference point 4 of the elastic block have their maximum value. The elastic block enters the curved part for the second time at \( t = 5.6 \text{ s} \). Due to the effect of friction force it doesn’t reach again the left inclined part any more and its reciprocal motion will continue in the...
curved part till the end of simulation. This time, \( t = 10 \text{ s} \), is not sufficient to bring the elastic block in sticking contact and, therefore, until the end of simulation, it will have sliding contact.

Figures 4c and 4d show translational and rotational velocities of the elastic block. The above-mentioned motion of the elastic block can also be seen from these figures. The trajectory of movement of the elastic block is also depicted in Figure 4e and the elastic coordinates are shown in Figure 4f. In the second simulation, one can see due to the larger value of the friction
coefficient, $\mu = 0.2$, that the elastic block will come to sticking contact.

As shown in Figure 5, due to the larger friction the elastic block reaches the curved part later than in the first case, namely at $t = 2.3\ s$. In this case, according to Figure 5b it reaches the lowest point of the curved path at $t = 3\ s$ but it cannot enter the right inclined part since $\theta$ doesn’t reach the constant value of $0.523\ rad$ which is the inclination angle of the right inclined part. Instead, it starts to move in the opposite direction at $t = 4\ s$. This point can be seen from...
Figures 5a and 5b. In the end, it comes to rest somewhere on the curved part at \( t = 7.6 \, s \) and at this time sliding contact switches to sticking one and the block will not have any movement after this time.

Figures 5c and 5d show the translational and rotational velocities of the elastic block for this case. The trajectory of movement of the elastic block and the elastic coordinates are shown in Figures 5e and 5f. It can be seen that even in the sticking phase there are still some elastic vibrations of the block.

### 4.2 Example 2: Constrained elastic rectangular block on a curved rigid foundation

In the second example, Figures 6, the contact of a constrained elastic block on a rigid half-circular foundation with radius \( r = 10 \, m \) is investigated. This example is simulated for two different constraints. In the first simulation, the constraint is supposed to keep the center point of the rectangular block in a constant distance from the center of the half-circular rigid foundation and in the second simulation the constraint has to keep the center point of the elastic block in a half-ellipse path according to the figure. It applies a constraint force which is considered in Eq. (11) by the parameter \( \lambda_c \). The second simulation is done in order to force the block to deform much more so that we can investigate the effect of deformations in the amount of contact forces. In other words, one can find out that how considering deformations will affect the contact forces. As one will see, as the rigidity of the elastic block is increased by increasing Young’s modulus, the amount of contact forces will approach a constant value.

The elastic block has the same material parameters as in the previous example, but the friction coefficient \( \mu = 0.5 \) is chosen. Some results of the first simulation are shown in Figure 7. The first and the second parts of Figure 7a and 7b show \( x \) and \( y \) positions and the orientation \( \theta \) of the reference coordinate of the elastic block over the time. From these figures, it can easily be seen that the block slides on the foundation until \( t = 8.2 \, s \) and at this time, sticking contact
appears and the block comes to a rest.

In Figure 8, for the second constraint the amount of contact force of node 1 in \( y \) direction when it passes through the lowest point of the rigid foundation (at \( x = 0 \) and \( y = -10 \text{ m} \)) for different values of Young’s modulus \( E \) is depicted. According to this figure, by increasing the value of \( E \), the contact force will increase too. The slope of these changes for the soft elastic block is decreasing and approaches zero for the stiff elastic block. Therefore, from this figure the importance and necessity of considering the deformations become clear and it can be seen that for such situations, the elastic contact modeling has to be used instead of rigid contact modeling.

5 CONCLUSION

In this paper, a formulation leading to linear complementarity problems for continual contact of deformable bodies was presented.

The procedure of this formulation started by formulating the kinematics of an arbitrary contact point \( k \) for an arbitrary pair of contacting bodies \( i \) and \( j \). In doing so, relative accelerations
of the contact point in normal and tangential directions are used. The effect of deformabilities were taken into account during the formulation of these accelerations.

In the next step, deformable bodies are modeled by utilizing the well-known moving frame of reference approach, see [11], [12], which is frequently used to generate the equations of motion of deformable bodies in multibody systems. In this approach, it is supposed that movement of deformable bodies consists of the rigid and the elastic parts and a small deformation is considered around a large rigid body motion. In the last step of this formulation, the complementarity form of the equations of motion was constructed by following the same procedure as described in [6] for rigid body contact. In order to compare the presented formulation with the formulation for rigid bodies, we started from the obtained relationships for relative accelerations of the contact point $k$ and then by ignoring the elastic coordinates, we reached the relative accelerations of the rigid case which is known from [6] using another notation.

For validation of the described procedure, the presented formulation was examined by two examples. In these examples, continual contact of a guided and a non-guided elastic block on the rigid foundation is investigated and at the end, the results obtained from simulation of these examples were given. In order to handle the case of impact of deformable bodies, this formulation must be extended which is current work of the authors.

REFERENCES


